Since $\binom{x}{m}$ is a polynomial of degree m in x, f_z is a polynomial of degree d.

Also solved by B. M. Ábrego, S. Amghibech (Canada), D. Beckwith, R. Chapman (U. K.), Y. Dumont (France), H. Gould, T. Hermann, J. H. Lindsey II, O. P. Lossers (Netherlands), N. C. Singer, A. Stadler, J. H. Steelman, R. Stong, L. Zhou, BSI Problems Group (Germany), GCHQ Problem Group (U. K.), and the proposer.

The Envelope, Please

10990 [2003, 59]. *Proposed by Rick Mabry, LSUS, Shreveport, LA.* The $n+1$ *Bernstein polynomials* of degree n are defined by

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \quad (0 \le k \le n).$$

When all $n+1$ polynomials are plotted on the same graph for large fixed n over the interval $0 \le x \le 1$, an 'upper envelope' begins to be seen. [A figure was given which is here omitted.] Let $\beta(x) = \lim_{n\to\infty} \sqrt{n} \max_{0\le k\le n} b_{n,k}(x)$. Find a closed-form expression for $\beta(x)$.

Bernardo M. Ábrego, California State University, Northridge, CA. The closed-form expression is $\beta(x) = 1/\sqrt{2\pi x(1-x)}$. Let $0 < x < 1$ and $1 \le k \le n$, and observe that

$$\frac{b_{n,k}(x)}{b_{n,k-1}(x)} = \frac{x(n-k+1)}{(1-x)k} = 1 + \frac{(n+1)x-k}{k(1-x)}.$$

Thus if $k < x(n+1)$ then $b_{n,k}(x) > b_{n,k-1}(x)$, and if $k > x(n+1)$ then $b_{n,k}(x) < b_{n,k-1}(x)$. Hence, if $m = m(n)$ is the unique integer such that

$$x(n+1) - 1 < m \le x(n+1), \tag{1}$$

then $\max_{0\le k\le n} b_{n,k}(x) = b_{n,m}(x)$, and $\beta(x) = \lim_{n\to\infty} \sqrt{n} b_{n,m}(x)$.

By Stirling's formula ($n! \sim \sqrt{2\pi n}(n/e)^n$) and the fact that $m \sim xn$ as $n \to \infty$, we have

$$\binom{n}{m} \sim \frac{n^n\sqrt{n}}{m^m(n-m)^{n-m}\sqrt{2\pi m(n-m)}},$$

which leads to

$$\lim_{n\to\infty} \sqrt{n} b_{n,m}(x) = \lim_{n\to\infty} \frac{n^{n+1}x^m(1-x)^{n-m}}{m^m(n-m)^{n-m}\sqrt{2\pi m(n-m)}}.$$

Now (1) is equivalent to $m/(n+1) < x \le (m+1)/(n+1)$, so

$$\frac{n+1}{\sqrt{2\pi(m+1)(n-m+1)}} < \frac{1}{\sqrt{2\pi x(1-x)}} < \frac{n+1}{\sqrt{2\pi m(n-m)}}$$

and

$$\frac{n^{n+1}(n+1-m)^{n-m}}{(n+1)^n(n-m)^{n-m}} < \frac{n^{n+1}x^m(1-x)^{n-m}}{m^m(n-m)^{n-m}} < \frac{n^{n+1}(m+1)^m}{(n+1)^n m^m}.$$

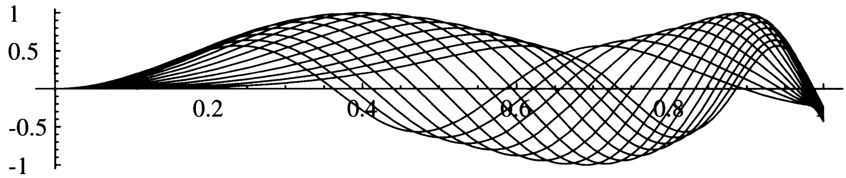
Thus $1/\sqrt{2\pi m(n-m)} \sim 1/((n+1)\sqrt{2\pi x(1-x)})$, hence

$$\lim_{n\to\infty} \frac{\left(1+\frac{1}{n-m}\right)^{n-m}}{\left(1+\frac{1}{n}\right)^{n+1}\sqrt{2\pi x(1-x)}} \le \lim_{n\to\infty} \sqrt{n} b_{n,m}(x) \le \lim_{n\to\infty} \frac{\left(1+\frac{1}{m}\right)^m}{\left(1+\frac{1}{n}\right)^{n+1}\sqrt{2\pi x(1-x)}}.$$

Finally, since $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$, we have $\beta(x) = 1/\sqrt{2\pi x(1-x)}$.

Editorial comment. The proposer notes a tempting but fallacious line of reasoning (adopted by several would-be solvers): Maximize $b_{n,k}(x)$ by setting $x = k/n$, forming the parametric curves $(k/n, \sqrt{n}b_{n,k}(k/n))$. Then let n go to ∞. However, this does not yield any information about envelopes. The proposer gives as an example the curves

$$y = \frac{\sin(\pi((1-c)x + cx^2))^2}{1+c^2} \qquad (x \in [0,1], c \in (-1,1)),$$



in which the relative maxima do not trace out the envelope. On the other hand, one can deduce the desired result from standard facts in probability theory such as the de Moivre-Laplace local limit theorem; several solvers referenced Feller's *An Introduction to Probability Theory and Its Applications*, third edition, chapter 7.

Also solved by S. Amghibech (Canada), R. Chapman (U. K.), D. Donini (Italy), Y. Dumont (France), T. Hermann, J. Kuplinsky, E. Lee, J.H. Lindsey II, O.P. Lossers (Netherlands), A. Nijenhuis, M. Pinsky, N.C. Singer, A. Stadler (Switzerland), R. Stephens, R. Stong, E.I. Verriest, L. Zhou, BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

A Nonholomorphic Map of the Disk

10991 [2003, 155]. *Proposed by Raymond Mortini, Départment de Mathematiques, Université de Metz, Ile du Saucy, France.* For complex $a, z \in \mathbb{D} = \{s\colon |s| < 1\}$, let $F(a,z) = \frac{a+z}{1+\bar{a}z}$ be a map of $\mathbb{D}$ onto $\mathbb{D}$. Let ρ denote the pseudohyperbolic distance, defined by $\rho(a,b) = \left|\frac{a-b}{1-\bar{a}b}\right|$.
(**a**) Prove that there exists a function $C\colon \mathbb{D} \to \mathbb{R}^+$ so that $\rho(F(a,z), F(b,z)) \le C(z)\rho(a,b)$ for every $a, b, z \in \mathbb{D}$.
(**b**) Find the minimal value of $C(z)$ for which this bound holds.

Solution by S. Amghibech, Québec, Canada. First note two identities:

$$\left|1 - \overline{F(a,z)}F(b,z)\right|^2 = \left(1 - |F(a,z)|^2\right)\left(1 - |F(b,z)|^2\right) + \left|F(a,z) - F(b,z)\right|^2,$$

$$\left(1 - |F(a,z)|^2\right)\left|1 + \bar{a}z\right|^2 = \left(1 - |a|^2\right)\left(1 - |z|^2\right).$$

From these we obtain

$$\rho\left(F(a,z), F(b,z)\right)^2 = \frac{Y}{\left(1-\rho(a,b)^2\right)\left|1-\bar{a}b\right|^2\left(1-|z|^2\right)^2 + Y},$$

where $Y = |(1+\bar{b}z)(a+z) - (1+\bar{a}z)(b+z)|^2$. Since

$$\frac{Y}{|a-b|^2} = \left|1 + \bar{a}z - \frac{\bar{a}-\bar{b}}{a-b}\left(az+z^2\right)\right|^2 \le \left(1+|z|+|z|+|z|^2\right)^2 = \left(1+|z|\right)^4,$$