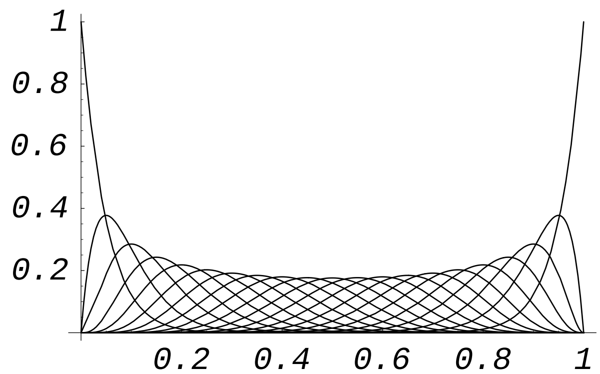
Bernstein envelopes

Proposed by Rick Mabry, LSUS, Shreveport, LA, rmabry@pilot.lsus.edu.

The $n+1$ Bernstein polynomials of degree n are defined by

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, 1, 2, \ldots, n.$$

When all $n+1$ polynomials are plotted on the same graph for a large fixed n and $0 \le x \le 1$, an "upper envelope" begins to emerge.



$y = b_{n,k}(x), \quad 0 \le k \le n$ for $n = 20$.

In fact, for suitable constants B_n (depending only on n), the $n+1$ plots of

$$y = B_n b_{n,k}(x), \quad 0 \le x \le 1, \quad k = 0, 1, 2, \ldots, n$$

appear to approach a single (i.e., independent of n) fixed envelope $y = \beta(x)$. Find suitable B_n and $\beta(x)$ (the choices are not unique).

[Note to editor: The pictures herein are not necessary. They are included only as enticements.]

Solution. We'll show that for $B_n = \sqrt{n}$ the limiting envelope is

$$y = \beta(x) = \frac{1}{\sqrt{2\pi x(1-x)}}.$$

(Note: the choice for B_n is not unique; any constants asymptotic to $\sqrt{n}$ will do. One can also multiply these B_n and $\beta(x)$ by arbitrary positive constants and still satisfy the statement of the problem.)

Method 1. Fix n and consider k to vary continuously in $(0, n)$ (using gamma functions in place of factorials), obtaining an infinite family of curves which vary smoothly with k. It is convenient to let $c = k/n$ and deal with the family of curves

$$\begin{aligned} y &= F(x,c) \equiv B_n b_{n,nc}(x) \\ &= \frac{\sqrt{n}\,\Gamma(n+1)\,x^{nc}(1-x)^{n(1-c)}}{\Gamma(nc+1)\Gamma(n(1-c)+1)}, \quad 0 \le x \le 1, \quad 0 < c < 1. \end{aligned}$$

We now apply classical methods (see, e.g., [**1**]), according to which the envelope is the solution to

$$\frac{\partial F}{\partial c} = 0. \tag{1}$$

Letting ψ denote the digamma function,

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)},$$

simplifying (1) yields

$$\log\left(\frac{1-x}{x}\right) = \psi(n(1-c)+1) - \psi(nc+1).$$

Letting $x_n(c)$ denote the solution (in x) to this equation we obtain that

$$x_n(c) = \frac{1}{1 + e^{\psi(n(1-c)+1) - \psi(nc+1)}}. \tag{2}$$

The envelope of curves, for a fixed n, is then given by the parameterized curve

$$(x_n(c), y_n(c)), \quad 0 < c < 1,$$

where $y_n(c) = F(x_n(c), c)$. To find the limiting envelope as $n \to \infty$ we use the fact that

$$\psi(z) = \ln z - \frac{1}{2z} + \mathcal{O}\left(\frac{1}{z^2}\right) \tag{3}$$

(see, e.g., entry 6.3.18 or 6.3.21 in **[2]**) and we have

$$\lim_{n\to\infty} \left(\psi(n(1-c)+1) - \psi(nc+1)\right) = \ln\left(\frac{1-c}{c}\right),$$

which, using (2), implies that $x_n(c) \to c$ as $n \to \infty$. Noting that the quantities nc and $n(1-c)$ go to ∞ as $n \to \infty$, and we can we apply the usual asymptotic properties of the gamma function (i.e., Stirling's formula) to get

$$\begin{aligned}\lim_{n\to\infty} y_n(c) &= \lim_{n\to\infty} F(x_n(c), c) = \lim_{n\to\infty} \frac{\sqrt{n}\,\Gamma(n+1)\; c^{nc}(1-c)^{n(1-c)}}{\Gamma(nc+1)\Gamma(n(1-c)+1)} \\ &= \frac{1}{\sqrt{2\pi c(1-c)}}.\end{aligned}$$

We have shown that

$$\lim_{n\to\infty} (x_n(c), y_n(c)) = \left(c, \frac{1}{\sqrt{2\pi c(1-c)}}\right),$$

which proves the claim.

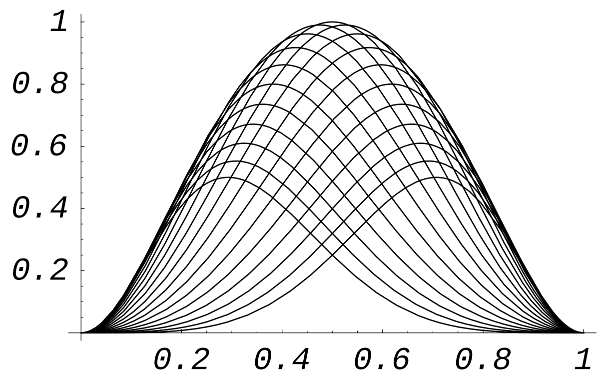
Method 2. A sketch of the curves in question shows that, for a fixed n, the "peaks" of the curves $y = \sqrt{n}\; b_{n,k}(x)$ appear to approach a single smooth curve. Following this notion, and using elementary properties of binomial coefficients, it is easy to show that each of the functions $b_{n,k}(x)$ has a single relative maximum in the interval $(0, 1)$ when $x = k/n$. The apparent envelope can be found by first treating k as a continuous variable and forming the parameterized (in k) curves (for each n)

$$H_n(k) \equiv \left(k/n, \sqrt{n}\; b_{n,k}(k/n)\right),$$

and then letting $n \to \infty$. The trouble is that this reasoning is completely bogus! Such relative maxima don't, in general, have anything to do with envelopes of curves. Explicit illustrations of this are easy to construct. For example, the maxima of the family of curves

$$y = h(c,x) = \frac{\sin(\pi\ ((1-c)\ x + c\,x^2))^2}{1+c^2}, \quad x \in [0,1], c \in (-1,1)$$

do not coincide with the envelope of the family.



[Dear editor: This second method of “solution” is, of course, quite unnecessary to mention unless a significant number of solvers employ it!]

[**1**] Yates, R.C., *Curves and Their Properties*, NCTM, Washington, D.C., 1974, ISBN: 087-35303-9X.

[**2**] Abramowitz, M., Stegun, I.A. (editors), *Handbook of Mathematical Functions*, Dover Publications, Inc., New York, 1972, ISBN: 486-61272-4.