Proposed by Rick Mabry, LSU-Shreveport, and Paul Deiermann, Lindenwood University

Let $r(\theta) = 1 + b\cos(\theta)$, where $0 < b \leq 1$, describe a limaçon in polar coordinates. Find the smallest rectangle of the form $[x_1, x_2] \times [y_1, y_2]$ which contains all of these graphs. (To achieve a nice computer animation, e.g., using *Mathematica*, of this one parameter family of polar graphs, one would prefer a fixed rectangular viewing window that contains all of the graphs. We seek the smallest such window.)

Solution (I) by Paul Deiermann, Lindenwood University and Rick Mabry, LSU-Shreveport

It is trivial to show that $[x_1, x_2] = [-1, 2]$, and also it is obvious by symmetry that $[y_1, y_2]$ has the form $[-c, c]$ for some $c > 0$. By this symmetry we need only consider $0 \leq \theta \leq \pi$. For any fixed b, we seek the maximum y-value, where $y = y(\theta) = r(\theta)\sin\theta$. Setting $y'(\theta) = 0$ to find the critical angle, we could solve explicitly and obtain

$$\theta = \theta(b) = \arccos\left(\frac{-1 + \sqrt{1 + 8b^2}}{4b}\right),$$

where the positive root is chosen for reasons that will soon be clear.

Instead of using the explicit form of the critical angles, the problem can be solved implicitly. The function $f(\theta) = \cos\theta/\cos 2\theta$, for $0 \leq \theta \leq \pi$, is strictly increasing on its continuous components, with vertical asymptotes at $\theta = \pi/4$ and $\theta = 3\pi/4$. By inspection, there is a unique intersection point of the horizontal line $y = -b$ with the graph of $y = f(\theta)$. The critical angle $\theta(b)$ for each b is the abscissa of this point. It is easy to see that $\theta(b)$ is a strictly decreasing function of b and that $\arccos(1/2) \leq \theta(b) < \pi/2$. (These bounds are the reason for choosing the positive root in the explicit formula for $\theta(b)$.)

Since $y(0) = y(\pi) = 0$ and $0 < y(\theta)$ for $0 < \theta < \pi$, the critical angle $\theta(b)$ must be a global maximum for $y(\theta)$ for each fixed b. The problem is now reduced to maximizing $y(\theta(b))$ over all b, with $0 < b \leq 1$.

The critical angle satisfies the equation

$$\cos(\theta(b)) + b\cos(2\theta(b)) = 0. \tag{$\star$}$$

An implicit differentiation reveals that

$$\theta'(b) = \frac{\cos(2\theta(b))}{\sin(\theta(b)) + 2b\sin(2\theta(b))}.$$

Note that this derivative is always negative, since $\theta(b)$ is strictly decreasing. In particular, it can be directly verified that the denominator is never zero.

A calculation combined with $(\star)$ demonstrates that

$$\begin{aligned} y'(\theta(b)) &= [\cos(\theta(b)) + b\cos(2\theta(b))]\,\theta'(b) + \sin(\theta(b))\cos(\theta(b)) \\ &= 0 + \sin(\theta(b))\cos(\theta(b)), \end{aligned}$$

which is clearly positive since $0 < \cos(\theta(b)) \leq 1/2$ and $0 \leq \theta \leq \pi$ for all θ. As a consequence, the function $y(\theta(b))$ is a strictly increasing function of b, which implies that the maximum is attained at $b = 1$. Since $\cos(\theta(1)) = 1/2$, trivially, $\sin(\theta(1)) = \sqrt{3}/2$, hence the maximum of the y-coordinates of the limaçons is $y(\theta(1)) = 3\sqrt{3}/4$. †

Solution (II) by Rick Mabry, LSU-Shreveport and Paul Deiermann, Lindenwood University

It is easy to see that for each fixed b, the maximum y value occurs when

$$\theta = \theta(b) = \arccos\left(\frac{-1+\sqrt{1+8b^2}}{4b}\right),$$

as noted in (I). Letting $g(b)$ denote the y-value for this angle, namely, $g(b) = y(\theta(b))$, the graph appears to be increasing for $0 < b \leq 1$. If so, our maximum is at $g(1) = 3\sqrt{3}/4$. We show directly that g' is indeed positive for all $0 < b \leq 1$. Indeed, a routine (albeit tedious) calculation shows that

$$g'(b) = \frac{-1-6b^2+16b^4+\left(1+2b^2\right)\sqrt{1+8b^2}}{4b^2\sqrt{2+16b^2}\sqrt{-1+4b^2+\sqrt{1+8b^2}}}. \qquad (\star\star)$$

The denominator is easily seen to be defined and positive for $0 < b \leq 1$, and the numerator is positive if and only if

$$1+6b^2-16b^4 \leq \left(1+2b^2\right)\sqrt{1+8b^2},$$

which, after factoring the left-hand side, reduces to

$$\left(1-2b^2\right)\sqrt{1+8b^2} \leq \left(1+2b^2\right).$$

Then, since the right-hand side of the above is nonnegative, the above inequality is true iff the resulting inequality remains true when both sides are squared. What results is the inequality

$$\left(1-2\,b^2\right)^2\left(1+8\,b^2\right)-\left(1+2\,b^2\right)^2 \leq 0,$$

which easily expands to

$$32\left(b^6-b^4\right) \leq 0,$$

which is clearly true for $0 < b \leq 1$. ††

Finally, a somewhat less algebraic approach is possible by applying the inequality

$$\sqrt{1+x} \geq 1 + x/2 - x^2/8$$

(which is esily proved with the Mean Value Theorem for all $x \geq 0$) to the numerator of $(\star\star)$.