Crosscut Convex Quadrilaterals

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In all that follows, $\mathcal{A}$ is an arbitrary convex plane quadrilateral; let's call such a polygon a *quad*. We let the vertices of $\mathcal{A}$ be given cyclically by the four-tuple (A_0, A_1, A_2, A_3) and name a midpoint M_i on each segment (A_i, A_{i+1}), taking all subscripts modulo 4, as in FIGURE 1. We then *crosscut* quad $\mathcal{A}$ by drawing *medians* (A_i, M_{i+2}). These medians intersect one another at the vertices of a new "inner quad" $\mathcal{B}$ in the interior of the "outer quad" $\mathcal{A}$.

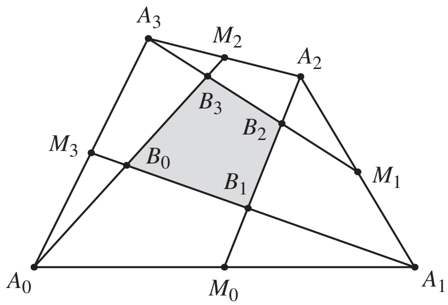

Figure 1 Crosscut quad $\mathcal{A}$ with shaded inner quad $\mathcal{B}$

Our results are inspired by the well known, pretty result ([**1**, p. 49], [**8**, p. 22]) that if $\mathcal{A}$ is a square, then

$$|\mathcal{A}| = 5\,|\mathcal{B}|\,, \tag{1}$$

where $|\cdot|$ denotes area. (See FIGURE 2. In this case it is clear that $\mathcal{B}$ is also a square.)

It follows from familiar facts about shear transformations that (1) remains true when $\mathcal{A}$ is a parallelogram. That (1) does not hold in general is easily seen by letting one

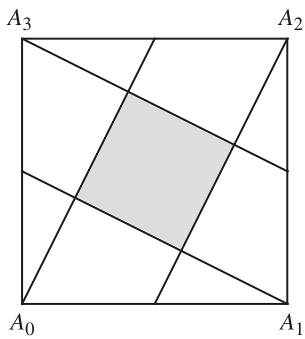

Figure 2 The classic crosscut square

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vertex of $\mathcal{A}$ approach another, in which case the shape of $\mathcal{A}$ approaches that of a triangle. In such a limiting case we get

$$|\mathcal{A}| = 6\,|\mathcal{B}|\,, \tag{2}$$

a fact that we leave as an easy exercise for the reader. (See FIGURE 3, where our four medians have coalesced into two medians of a triangle, these meeting at the triangle's centroid.) We will continue to refer to such a figure as a quad, albeit a *degenerate* one, and this will be the only type of degeneracy we need to consider—two vertices of a quad merging to form a nondegenerate triangle. Otherwise, a (nondegenerate) quad shall be convex with four interior angles strictly between zero and 180 degrees.

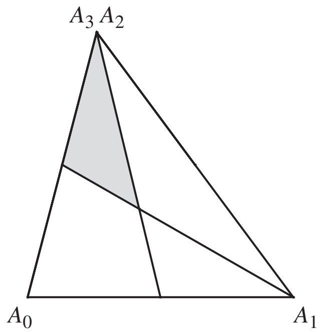


Figure 3 A degenerate case of two coincident vertices

It will be shown presently that the general case lies between (1) and (2). Actually, we prove a bit more:

THEOREM 1. *For an arbitrary outer quad $\mathcal{A}$, the following properties hold.*

(a) *The inner quadrilateral $\mathcal{B}$ is a quad and*

$$5\,|\mathcal{B}| \le |\mathcal{A}| \le 6\,|\mathcal{B}|\,. \tag{3}$$

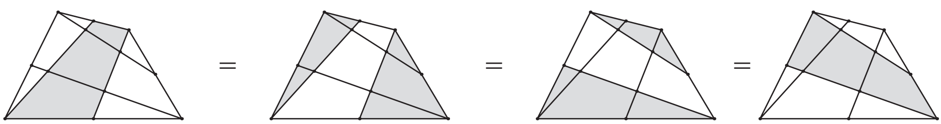
(b) $|\mathcal{A}| = 5\,|\mathcal{B}|$ *if and only if $\mathcal{B}$ is a trapezoid.*

(c) $|\mathcal{A}| = 6\,|\mathcal{B}|$ *if and only if $\mathcal{A}$ is a degenerate quad with two coincident vertices.*

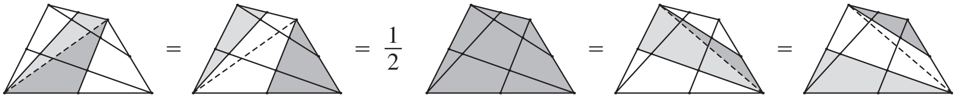
Theorem 1 has gathered dust since 2000 (or earlier—see the Epilogue) while the author occasionally tried, to no avail, to find a "Proof Without Words" (PWW) of (3) or some other *visual proof*, as is done in [**1**] and [**8**] for the case of a square. Later in this note a fairly visual proof will be given that $\mathcal{B}$ being a trapezoid implies that $|\mathcal{A}| = 5\,|\mathcal{B}|$, but that is far from the entire theorem. In part, the purpose of this note is to open this challenge to a wider audience.

In the meantime, we do have a few other visual propositions to offer. In the following three results and their proofs (WW), the values being added, subtracted, and equated are the areas of the shaded regions of an arbitrary (fixed) quad $\mathcal{A}$.

PROPOSITION 1. (STRIPS EQUAL HALF OF QUAD)



PWW of Proposition 1.



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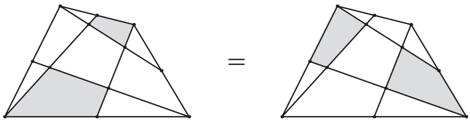
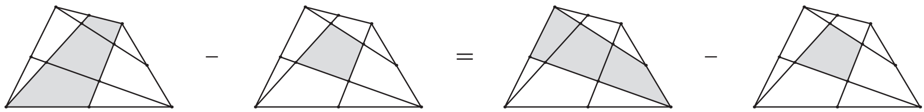
PROPOSITION 2. (PAIRS OF OPPOSING FLAPS ARE EQUAL)



PWW of Proposition 2.



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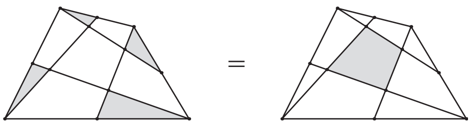
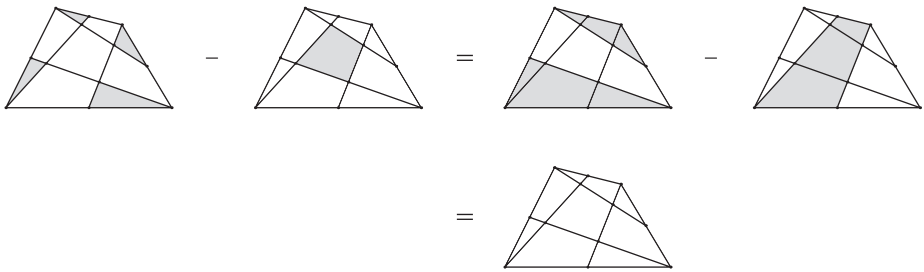
PROPOSITION 3. (CORNERS EQUAL INNER QUAD)



PWW of Proposition 3.



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The greatest visual appeal in this note might be concentrated in Proposition 3. It is a challenge (left to the readers and unfulfilled by the author) to prove some of the other facts in more visual ways, especially Theorem 1. Meanwhile we apply some easy vector-based methods that have some sneaky appeal of their own. Complex variables can be used to the same effect.

Notation, convention, and basic calculation First let's set some notation. Given a sequence $P_0, P_1, \ldots, P_n$ of points in $\mathbb{R}^2$, we let $(P_0, P_1, \ldots, P_n)$ denote either the ordered tuple of points or the polygon formed by taking the points in order. We shall assume that when $(P_0, P_1, \ldots, P_n)$ is given, *the sequence is oriented positively* (counterclockwise) in the plane, with this one exception: If the tuple is a pair, then we consider it *directed* so that we may use it as a vector as well as a line segment, the context making clear which. So, for instance, we write $\mathcal{A} = (A_0, A_1, A_2, A_3)$, and likewise we take $\mathcal{B} = (B_0, B_1, B_2, B_3)$ for our inner quad, where $B_i = (A_i, M_{i+2}) \cap (A_{i+1}, M_{i+3})$. We let $|(P_0, P_1, \ldots, P_n)|$ denote the area of the polygon $(P_0, P_1, \ldots, P_n)$, but $|(P, Q)|$

will denote the length of a segment or vector (P, Q). Likewise, $|\mathbf{w}|$ will denote the length of any vector $\mathbf{w}$. As usual, a point P is identified with the vector (O, P) via the usual canonical identification once an origin O is selected.

Areas are calculated using the magnitude of the cross product. It will be convenient to abuse notation slightly and, for $\mathbf{x} = \langle x_1, x_2 \rangle$ and $\mathbf{y} = \langle y_1, y_2 \rangle$, set $\mathbf{x} \times \mathbf{y} = x_1 y_2 - x_2 y_1$.

The area of any triangle is then $|(\mathbf{x}, \mathbf{y}, \mathbf{z})| = \frac{1}{2}(\mathbf{y} - \mathbf{x}) \times (\mathbf{z} - \mathbf{x})$, which is positive because of our orientation convention mentioned earlier.

For arbitrary P_0, P_1, P_2, P_3, the polygon formed from the sequence is a nondegenerate convex quadrilateral (and therefore also *simple*, that is, with no self-crossings) if and only if $(P_i, P_{i+1}) \times (P_{i+1}, P_{i+2})$ has constant positive or constant negative sign for each i (mod 4). (An analogous statement cannot be made for convex n-gons with $n \geq 5$—consider the pentagram, or, if the fewest crossings are desired, have a look at a "foxagon" like $(\langle 0, 0\rangle, \langle 3, 9\rangle, \langle -1, 5\rangle, \langle 1, 5\rangle, \langle -3, 9\rangle)$.) We will appeal to cross products to verify the convexity of a certain octagon that arises in our figures.

Diagonals rule Any two fixed, independent vectors $\mathbf{u}$ and $\mathbf{v}$ in the plane correspond to the diagonals of infinitely many different quadrilaterals, as is illustrated in FIGURE 4, where the diagonals of the quadrilaterals shown generate identical vectors. Clearly, such quadrilaterals need not be quads—shown in that figure are a nonconvex and a nonsimple quadrilateral (neither are quads) and two quads, one being a parallelogram. To ensure convexity, it is necessary and sufficient that the diagonals intersect each other. We accomplish this in the following way. Let the origin O be the intersection of the diagonals, and for scalars s and t, set

$$(A_0, A_1, A_2, A_3) = ((1-s)\mathbf{u}, (1-t)\mathbf{v}, -s\,\mathbf{u}, -t\,\mathbf{v}), \tag{4}$$

as in FIGURE 5. When s and t in $[0, 1]$ the quadrilateral will be convex; it will be a quad when s and t are in $(0, 1)$. The diagonals (A_0, A_2) and (A_1, A_3) have lengths $|\mathbf{u}|$ and $|\mathbf{v}|$, respectively.

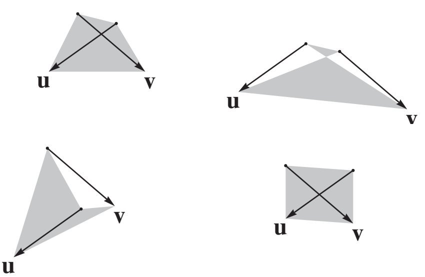

Figure 4 Two linearly independent vectors **u** and **v** form (nonuniquely) the diagonals of a quadrilateral

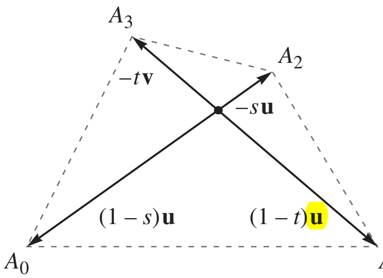

Figure 5 Splitting **u**, **v** with s, t to specify a unique quad

It is a simple matter to compute the area of any simple polygon by triangulation (partitioning the polygon into triangles). For a simple, positively oriented quadrilateral $\mathcal{Q} = (Q_0, Q_1, Q_2, Q_3)$ one may also use the familiar fact that $|\mathcal{Q}| = \frac{1}{2}(Q_0, Q_2) \times (Q_1, Q_3)$. It is then clear that in the case of our main quad, $|\mathcal{A}| = \frac{1}{2}\mathbf{u} \times \mathbf{v}$.

We exploit the fact that all of the vertices of the polygons that appear in the context of crosscut quads are linear combinations of $\mathbf{u}$ and $\mathbf{v}$, with rational functions of s, t

as coefficients. Therefore, since $\mathbf{u} \times \mathbf{u} = \mathbf{v} \times \mathbf{v} = 0$, all of our polygonal areas take the form $F(s, t)\, \mathbf{u} \times \mathbf{v}$ for some rational function F of s, t. All our results derive from $F(s, t)$, as we could scale our figures to have $\mathbf{u} \times \mathbf{v} = 1$.

Proof of Theorem 1. First note that $\mathcal{B}$ is indeed convex, being the intersection of two (clearly) convex sets. To compute $|\mathcal{B}|$, we first find expressions for the points B_i. First, we have that $B_0 = (A_0, M_2) \cap (A_1, M_3)$, where $M_i = \frac{1}{2}(A_i + A_{i+1})$. There are then scalars $q, r \in (0, 1)$ for which

$$B_0 = A_0 + q(M_2 - A_0) = A_1 + r(M_3 - A_1).$$

This implies that

$$\frac{q}{2}\big((A_2 - A_0) + (A_3 - A_0)\big) = A_1 - A_0 + \frac{r}{2}\big((A_3 - A_1) + (A_0 - A_1)\big),$$

hence, by (4),

$$\frac{q}{2}\big(-\mathbf{u} - t\,\mathbf{v} - (1 - s)\mathbf{u}\big) = (1 - t)\mathbf{v} - (1 - s)\mathbf{u} + \frac{r}{2}\big(-\mathbf{v} + (1 - s)\mathbf{u} - (1 - t)\mathbf{v}\big).$$

The linear independence of $\mathbf{u}$ and $\mathbf{v}$ allows us to separately equate their coefficients to solve for q and r, obtaining

$$q = 2(1 - s)/(4 - 2s - t) \quad \text{and} \quad r = 2(2 - s - t)/(4 - 2s - t).$$

(The most diligent of readers will pause to verify that both q and r lie in $(0, 1)$ when s and t do.) We can use q (say) to calculate

$$B_0 = A_0 + q(M_2 - A_0) = \frac{(1 - s)(2 - s - t)}{4 - 2s - t}\,\mathbf{u} - \frac{(1 - s)t}{4 - 2s - t}\,\mathbf{v}.$$

Once B_0 is obtained, the symmetry of our construction in FIGURE 5 can be used to compute the remaining B_i—our convention for indexing points ensures that $P_{i+1} = \langle -y(t, 1 - s),\ x(t, 1 - s)\rangle$ when $P_i = \langle x(s, t),\ y(s, t)\rangle$ is given.

We can now calculate the area of $\mathcal{B}$. We could use the formula for the area of a quadrilateral already mentioned, but instead we'll employ the result of Proposition 3, as we'll make use of the result later. Denoting by $\mathcal{C}_i$ the corner triangle (A_i, B_i, M_{i-1}), we easily compute

$$|\mathcal{C}_0| = |(A_0, B_0, M_3)| = \frac{(1 - s)t}{4(4 - 2s - t)}(\mathbf{u} \times \mathbf{v}). \tag{5}$$

Again, symmetry can be used to compute the remaining three corner areas from the first; generally, if $|\mathcal{R}_i| = g(s, t)$, then $|\mathcal{R}_{i+1}| = g(t, 1 - s)$. (The fact that $(-\mathbf{v}) \times \mathbf{u} = \mathbf{u} \times \mathbf{v}$ is used for this.)

Now we break out the algebra software (if we haven't already) and find, using $|\mathcal{B}| = |\mathcal{C}_0| + |\mathcal{C}_1| + |\mathcal{C}_2| + |\mathcal{C}_3|$, that

$$6\,|\mathcal{B}| - |\mathcal{A}| = \frac{5\left(2 - 3s + s^2 + 4s\,t - 2t^2\right)\left(4s + t - 2s^2 - 4s\,t + t^2\right)}{2(2 - s + 2t)(3 + s - 2t)(4 - 2s - t)(1 + 2s + t)}(\mathbf{u} \times \mathbf{v}). \tag{6}$$

It is now a routine exercise to show that for $s, t \in (0, 1)$, each of the factors above is positive. This shows that $6\,|\mathcal{B}| \geq |\mathcal{A}|$. Similarly, we can compute

$$|\mathcal{A}| - 5\,|\mathcal{B}| = \frac{(1 - 3s + t)^2(2 - s - 3t)^2}{2(2 - s + 2t)(3 + s - 2t)(4 - 2s - t)(1 + 2s + t)}(\mathbf{u} \times \mathbf{v}), \tag{7}$$

which is clearly nonnegative on $[0, 1] \times [0, 1]$. Part (a) of the theorem is now established.

Part (b) of the theorem is fairly easy in view of the fact that (7) shows that $|\mathcal{A}| - 5|\mathcal{B}| = 0$ if and only if $1 - 3s + t = 0$ or $2 - s - 3t = 0$. Consider the easily verified fact that

$$(B_0, B_1) \times (B_0, B_3) \\ = -\frac{(1-3s+t)\left(2s+t-s^2-st+t^2\right)\left(2+s-2t-s^2-st+t^2\right)}{(2-s+2t)(3+s-2t)(4-2s-t)(1+2s+t)}(\mathbf{u} \times \mathbf{v}).$$

None of the factors, other than $1 - 3s + t$, is ever zero for $s, t \in (0, 1)$; and it follows that for a crosscut quad $\mathcal{A}$, the segments (B_0, B_1) and (B_2, B_3) are parallel if and only if $1 - 3s + t = 0$. Similarly (after all, there is a symmetry at work here), one finds that (B_1, B_2) is parallel to (B_0, B_4) if and only if $2 - s - 3t = 0$. This establishes part (b). (We can also see from this that $\mathcal{B}$ is a parallelogram if and only if $1 - 3s + t = 0 = 2 - s - 3t$, which happens if and only if $s = t = 1/2$, which is in turn true if and only if $\mathcal{A}$ is a parallelogram.)

Finally, for part (c), it was probably noticed earlier that for all $s, t \in [0, 1]$, all the factors in the denominator of (6) are ≥ 1 and that the numerator is zero if and only if $\langle s, t\rangle$ is one of $\langle 0, 0\rangle$, $\langle 1, 0\rangle$, $\langle 0, 1\rangle$, or $\langle 1, 1\rangle$. It is clear from our construction that each of these four cases is equivalent to the (degenerate) situation of the merging of two vertices of $\mathcal{A}$. (For example, $A_2 = A_3$ when $\langle s, t\rangle = \langle 0, 0\rangle$, as in FIGURE 3.) ■

Diagonal triangles There are probably many amusing relationships lurking among the various pieces of our crosscut quad. As an example (found, as with the others, by messing around with *Geometer's Sketchpad*), our next theorem contains a cute result, whose visual proof, too, we abandon to the reader.

First a few easy preliminaries. The diagonals of a quad partition the quad into four triangles. With O being the intersection of the diagonals of $\mathcal{A}$ (the origin mentioned earlier), we define the *diagonal triangles* $\mathcal{D}_i = (A_i, O, A_{i-1})$. If the diagonals of $\mathcal{A}$ are added to the picture, as in FIGURE 6, it is clear that the points B_i must always lie within the respective $\mathcal{D}_i$. Incidentally, it is a simple exercise to show that the centroids D_i of the $\mathcal{D}_i$ form a parallelogram whose area is $(2/9)\,|\mathcal{A}|$. (Engel [**4**, prob. 69, sec. 12.3.1] shows that this result holds when O is *any* point in the quad.) Also known is that the products of the areas of opposing diagonal triangles $\mathcal{D}_i$ are equal. That is,

$$|\mathcal{D}_1|\,|\mathcal{D}_3| = |\mathcal{D}_0|\,|\mathcal{D}_2|\,, \tag{8}$$

which is easily checked by noting $|\mathcal{D}_0| = (1 - s)\,t/2$ and its cyclic counterparts. (A PWW lurks in FIGURE 5. In fact, (8) holds when O is any point on a diagonal of a quadrilateral $A_0A_1A_2A_3$, regardless of whether the quadrilateral is convex or simple. Cross products are superfluous.)

Corner triangles Another odd relationship involves ratios of areas of corner triangles to diagonal triangles. It may be unrelated to (8), but it has a similar flavor.

THEOREM 2. *For an arbitrary quad $\mathcal{A}$,*

$$\frac{|\mathcal{D}_0|}{|\mathcal{C}_0|} + \frac{|\mathcal{D}_2|}{|\mathcal{C}_2|} = \frac{|\mathcal{D}_1|}{|\mathcal{C}_1|} + \frac{|\mathcal{D}_3|}{|\mathcal{C}_3|} = 10.$$

Proof. Notice that the areas of the $\mathcal{D}_i$ are the numerators of the formulas we have for the areas of respective triangles $\mathcal{C}_i$ (see (5)). These cancel in each of the ratios

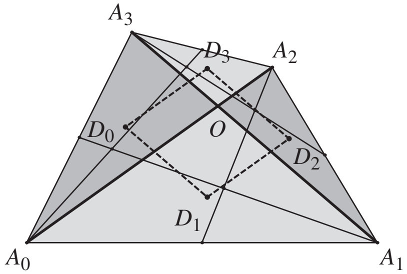

Figure 6 The $\mathcal{D}_i$, individually shaded, and the parallelogram of their centroids

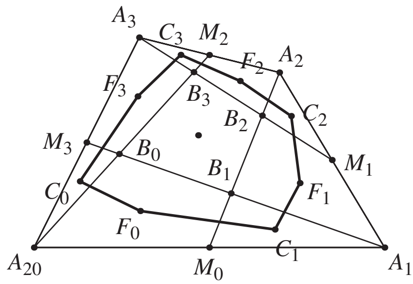

Figure 7 A convex octagon of centroids

$|\mathcal{D}_i| \,/\, |\mathcal{C}_i|$. Thus,

$$\frac{|\mathcal{D}_0|}{|\mathcal{C}_0|} + \frac{|\mathcal{D}_2|}{|\mathcal{C}_2|} = 2(4 - 2s - t) + 2(1 + 2s + t) = 10.$$

It is unnecessary to write a similar formula for the second sum, as it follows by symmetry. ■

A convex octagon In FIGURE 7, the centroids of the nine partitioned regions of our crosscut quad are shown. The outer eight form an octagon: Let $\mathcal{F}_i = (A_i, M_i, B_{i+1}, B_i)$, $i = 0, 1, 2, 3$, denote the flaps and let C_i and F_i denote the centroids of the corners $\mathcal{C}_i$ and the flaps $\mathcal{F}_i$, respectively.

THEOREM 3. *The octagon* $\mathcal{O} = (C_0, F_0, C_1, F_1, C_2, F_2, C_3, F_3)$ *is convex.*

All we offer toward a proof of the convexity is the suggestion already given concerning cross products. (The author's colleague Zsolt Lengvárszky has a nice visual proof.) It suffices to show that $(C_0, F_0) \times (F_0, C_1) > 0$ and $(F_0, C_1) \times (C_1, F_1) > 0$ for every choice of $s, t \in (0, 1)$. It turns out that, using our machinations here, the latter of these inequalities (which would appear from FIGURE 7 to be the more sensitive of the two) is fairly easy, as one gets a rational function of s, t with nice, positive polynomial factors. The former also factors into positive polynomial factors, although more work is involved and one of the factors is of degree five in s, t. We claim very tight bounds on the octagon's relative area:

$$1.888\cdots = \frac{270}{143} \le \frac{|\mathcal{A}|}{|\mathcal{O}|} \le \frac{216}{113} = 1.911\ldots,$$

where the lower bound occurs when $\mathcal{A}$ is a parallelogram, the upper when $\mathcal{A}$ becomes degenerate. (We provide no proof for this conjecture, but the author will attempt one if offered a sufficient cash incentive.)

A look at FIGURE 6 suggests that the centroid of each of the diagonal triangles D_i is inside the corresponding $\mathcal{F}_i$. This too is easily shown using cross products; it suffices to note that $(A_0, M_2) \times (A_0, D_0) = st/6 > 0$ and that D_0 lies on (O, M_3). The reader is challenged to give a nice visual proof of this fact, but it cannot be denied that using cross products is very quick indeed.

A visual proof for the trapezoid case Here now is the visually oriented proof, mentioned earlier, that $|\mathcal{A}| = 5\,|\mathcal{B}|$ when $\mathcal{B}$ is a trapezoid. (This is one direction of Theorem 1(b).) In FIGURE 8, triangle (M_1, A_2, A_3) is rotated 180° about M_1 forming (M_1, A_1, A_3'), and (M_3, A_0, A_1) is likewise rotated about M_3 to form (M_3, A_3, A_1').

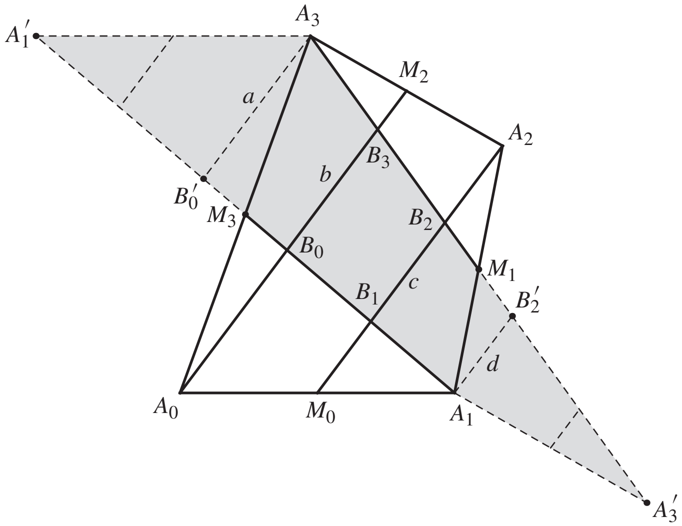

Figure 8 $\mathcal{B}$ is a trapezoid with $(B_1, B_2) \parallel (B_0, B_3)$

(For a point P, denote its rotated counterpart by P'.) Now let h (not annotated) be the perpendicular distance between segments (B_1, B_2) and (B_0, B_3), and let $a = |(B_0', A_3)|$, $b = |(B_0, B_3)|$, $c = |(B_1, B_2)|$, and $d = |(A_1, B_2')|$. Then it is evident that $a + d = b + c$, and we therefore have

$$\begin{aligned}|\mathcal{A}| &= \left|(B_0', A_3, A_1')\right| + \left|(B_0', A_1, B_2', A_3)\right| + \left|(A_1, A_3', B_2')\right| \\ &= \frac{1}{2}a(2h) + \frac{a+d}{2}(3h) + \frac{1}{2}d(2h) \\ &= 5\left(\frac{b+c}{2}h\right) = 5\,|\mathcal{B}|\,.\end{aligned}$$

If a condition on $\mathcal{A}$ itself is desired in order that $\mathcal{B}$ be a trapezoid, perhaps the following will satisfy. For fixed A_0, A_1, A_3, we will have $(M_0, A_2) \parallel (M_2, A_0)$ if and only if A_2 lies on the line joining the midpoint M_0 of (A_0, A_1) and the point that lies one-third the way from A_1 to A_3. This will force $(B_1, B_2) \parallel (B_0, B_3)$, as in FIGURE 8. Cyclically permute vertices for the remaining possibilities.

One can think of immediate variations and generalizations to the problems explored in this note. Note that there is a certain *chirality* or handedness in our choice of crosscutting; FIGURE 9 gives a version with alternate medians. We leave it as an exercise to prove that the areas of the inner quads of the two variations are equal (for a fixed outer quad) if and only if either the outer quad is a trapezoid or one of the diagonals of the outer quad bisects the other. Is there some visual proof of that?

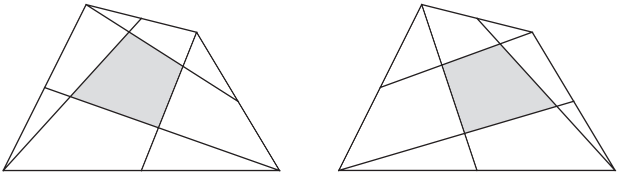

Figure 9 The original and alternate crosscut quads

To generalize, one can use other *cevians* in place of our medians, by letting $M_i = (1-r)A_i + r\,A_{i+1}$ for some fixed ratio r other than $r = 1/2$, as in FIGURE 10. It is then not difficult to establish a generalization to Theorem 1, in which the minimum and maximum for $|\mathcal{A}| \,/\, |\mathcal{B}|$ are replaced by $(r^2 - 2r + 2)/r^2$ and $(r^2 - r + 1)/r^3$, respectively. One can make more cuts to form an n-crosscut, skewed chessboard, as in FIGURE 11 (where $n = 4$). Obvious analogs of Theorems 1 and 2, and Propositions 2 and 3, hold for such multi-crosscuttings. For more generality, try $m \times n$-crosscut, skewed chessboards. An article by Hoehn [**6**] suggests further problems. As for a multi-crosscut generalization of Theorem 3, well, let's just say that what might seem an obvious generalization is not. (Not what? Not true or not obvious? The intrepid reader should venture forth.)

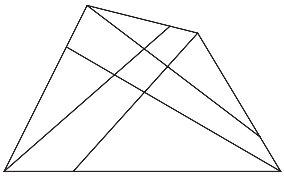

Figure 10 Crosscutting cevians with $r = 1/4$

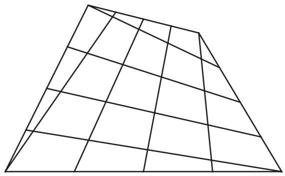

Figure 11 4-crosscut skewed chessboard

Epilogue It turns out that a form of Theorem 1 has appeared earlier. Our diligent referee found a reference to it [**3**, p. 132 (item 15.19)], which in turn names a problems column as a source [**9**]. For the same geometric configuration as ours, but using our notation, the statement in [**3**] gives the inequality $5\,|\mathcal{B}| \le |\mathcal{A}| < 6\,|\mathcal{B}|$ (so it doesn't count the degenerate case). However, [**3**] also states that $|\mathcal{A}| = 5\,|\mathcal{B}|$ "only for a parallelogram." This is incorrect, as we have shown that this equality holds if and only if $\mathcal{B}$ is a trapezoid, in which case neither $\mathcal{A}$ nor $\mathcal{B}$ need be a parallelogram. (It isn't clear in [**3**] whether $\mathcal{A}$ or $\mathcal{B}$ is intended as a parallelogram, but they turn out to be equivalent conditions.) Thus it was necessary to track down the original problem in [**9**] for comparison.

The problems column in question was in *Gazeta Matematică*, which is known to every Romanian mathematician, and is near and dear to most. It is one of the journals of *Societatea de Ştiinţe Matematice din România* (the Romanian Mathematical Society) [**2**]. The problem was posed by the eminent Romanian mathematician, Tiberiu Popoviciu (1906–1975), whose contributions to mathematics are too numerous to mention here. He is immortalized also by the Tiberiu Popoviciu Institute of Numerical Analysis, which he founded in 1957 (a short biography appears on the institute's website [**7**]).

Locating the problem, printed in 1943 (surely a difficult year), was not easy, as no library on the WorldCat® network has the volume. Fortunately, the entire collection is available in electronic form [**5**]. The generous help of Eugen Ionascu (Columbus State University) was enlisted, first to find someone who has access to the electronic format, and second for a translation into English (the text is Romanian). The translation revealed that, in *Gazeta*, Popoviciu gives the inequality as $5\,|\mathcal{B}| \le |\mathcal{A}| \le 6\,|\mathcal{B}|$ and challenges the reader to prove the inequality for all convex quadrilaterals $\mathcal{A}$ and to determine when equality holds. That seems to be the last mention of the problem. If there is a follow-up in later issues of *Gazeta Matematică*, it is hiding well. (It would be nice to know the solution intended by Popoviciu, which is likely more elegant than ours.)

Hearing of the search for the 1943 *Gazeta*, Aurel Stan (at The Ohio State University at Marion) had the following reaction. "*Gazeta Matematică* is one of the dearest things to my heart, although for many years I have not opened it, and I feel that I have betrayed it. It is one of the oldest journals in the world dedicated to challenging mathematical problems for middle and high school students. It has appeared without interruption since 1895." The Hungarian journal *Középiskolai Matematikai Lapok* (Mathematical Journal for High Schools) has a similar mission and has been published since 1894 except for a few years during WWII. Professor Stan continues: "Even during the two world wars the Romanian officers who had subscriptions had it [*Gazeta*] delivered to them in the military camps. It is probably the main reason why today so many foreign-born mathematicians in the United States are from Romania. We all grew up with it. Each month I waited for the newest issue." Professor Ionascu concurred with these sentiments, and described getting hooked on the journal in the seventh grade. Hungarian colleagues say similar things about *Matematikai Lapok*. Professor Stan mentions that even during the time of the Communist regime in Romania, there was a high level of mathematics and respect for mathematicians, adding, "We owe a big part of it to *Gazeta Matematică*."

Acknowledgments Certainly we all owe thanks to those earliest problem-solving journals and societies, and we can make partial payment on that debt by getting our students involved with their present-day counterparts. On that note, the author thanks his colleagues and the students in the Senior Seminar course at LSUS for their time spent listening to much of the above. Special thanks go to Roger B. Nelsen at Lewis & Clark College, who visited LSUS for a few short but enjoyable days during the spring semester of 2008, and whose many articles and books inspire so much visual thinking in mathematics. Thanks, too, to Aurel Stan and to Eugen Ionascu and his far-flung colleagues for help locating Popoviciu's problem.

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Summary A convex quadrilateral ABCD is "crosscut" by joining vertices A, B, C, and D to the midpoints of segments CD, DA, AB, and BC, respectively. Some relations among the areas of the resulting pieces are explored, both visually and analytically.

RICK MABRY'S professional and pleasurable pursuits (usually one and the same) seem mainly to involve subdividing time and/or space. As a drummer he cuts up time, and asks, Is it good when the pieces are reassembled and there is some left over? As a mathematician, he cuts up lines and planes and tries to be more careful, though he sometimes does cut corners (as in the present article) using *Mathematica* and Geometer's Sketchpad.