Selected answers to the Final Exam of Summer of '05...

Trigonometry | Summer 2005 | Name: ____________

The "No Calculators" Pages

Instructions: See front page for general instructions. Finish this page before going to the rest. You may not return to this page once you turn on your calculator.

N1.) (4 points) Find exact algebraic values for each of the following, where defined. Otherwise, write "undefined." 4

(a) $\cos 3\pi/4 \;= -\sqrt{2}/2$

(b) $\sin(-270°) \;= 1$

(c) $\cos(5\pi/6) \;= -\sqrt{3}/2$

(d) $\sec(-\pi/2)$ is undefined

(e) $\tan 240° \;= \sqrt{3}$

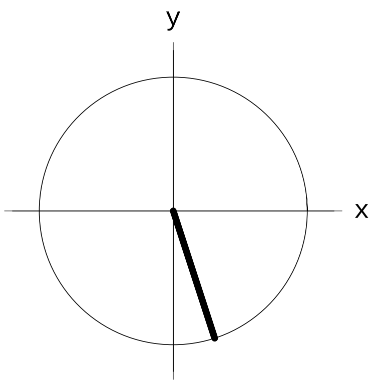
(f) $\sec 5\pi/4 \;= -\sqrt{2}$

(g) $\cot 5\pi/2 \;= 0$

(h) $\csc 210° \;= -2$

($\star$) Give the exact algebraic value for $\sin(126126126126126126126126126126126126126\pi/8)$
Dividing out multiples of 2π, we can see that the angle is coterminal with $-\pi/4$, so the sine is $-\sqrt{2}/2$

N2.) (4 points) An angle θ is shown below in standard position. Give an approximate value for $\cos\theta$, just by looking. No credit if you are off by more than two tenths! Better guesses get better scores. 8



It's around 0.3.

($\star$) Give an approximate numerical value for $\sin(183°)$. (The closer you get, the more points you get. You don't have to give a decimal — your answer can be "calculator ready" but it cannot have any trig formulas in it.)

It's close to $-\pi/60 \approx -.05$, and you need to get close for extra credit. A blind quess won't help much. The idea is that for small x, we know that $\sin x \approx x$, when x is in radians. You can adapt this idea to cases where, like here, x isn't given in radians and when it isn't close to zero but is still "low on the horizon".

N3.) (8 points) Spew some formulas. Write the standard identities for the following. 16

(a) $\sin(x+y)$

(b) $\cos(a-b)$

(c) $\tan(\alpha+\beta)$

(d) $\sin 2x$

(e) $\cos 2x$ (write all three!)

(f) $\sin(A/2)$

(g) $\tan(\theta/2)$ (write at least two)

(h) $\sin(\pi - x)$

N4.) (3 points) Write the *exact algebraic values* of the functions $\sin\theta$, $\cos\theta$ and $\tan\theta$ for the angle θ in standard position having the point $(\sqrt{2}, -3)$ on its terminal side. **19**

$\sin\theta = -3/\sqrt{11}$

$\cos\theta = \sqrt{2/11}$

$\tan\theta = -3/\sqrt{2}$

N5.) (3 points) Write **22**

(a) a reciprocal identity involving $\sin\theta$,

(b) a Pythagorean identity involving $\sec\theta$,

(c) a cofunction identity involving $\tan\theta$.

N6.) (4 points) Write the exact values of **26**

(a) $\arcsin(-1/2)$ $-\pi/6$

(b) $\arccos(-1/2)$ $2\pi/3$

Trigonometry Summer 2005
MATH 122-010 Name:

Final Exam

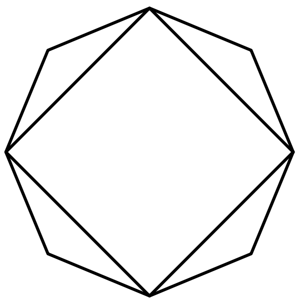
Instructions: Answer all problems correctly. Calculators are allowed (except on the "No Calculators Page") but *they must not be used to retrieve information or formulas*. Put final answers in the boxes provided. Each st⋆rred problem is extra credit, and each ⋆ is worth 5 points.

The phrase *exact algebraic values* appears throughout the test. Quantities such as $\sqrt{3}$, 5/3, etc., are exact algebraic values, as opposed to *numerical approximations*, such as 1.732, 1.666, etc., which are not.

1. (4 points) One fine morning, a tree standing exactly 15.3 feet tall casts a 35.7-foot shadow onto the level ground. What is the angle of elevation of the sun at that moment? 30

   $\arctan(15.3/35.7)$

2. (5 points) A regular octagon is circumscribed about a square whose area is 9 square inches, as in the figure below. Find the length of the perimeter of regular octagon. 35

   $24\sqrt{2}\sin(\pi/8)$

3. (4 points) A tourist on the ground views the Eiffel tower, which has a height of 986.0 feet. The top of the tower has an angle of elevation of 64.13°. The tourist backs directly away from the tower until the distance to the base of the tower is triple what it was previously. What is the new angle of elevation to the top of the tower? 39

   $\arctan(\frac{1}{3}\tan 64.13°)$

4. (6 points) A ship leaves port and travels for 32.0 miles with a bearing of S74.0°E. The ship then travels 17.0 miles with a bearing of N23.0°W. 45

   (a) How far is the ship from port?
   The distance is $D = \sqrt{x^2 + y^2}$, where $x = x_1 - x_2$ and $y = y_2 - y_1$, where $x_1 = 32 \sin 74°$, $y_1 = 32 \cos 74°$, $x_2 = 17 \sin 23°$ and $y_2 = 17 \cos 23°$. Some people used the law of cosines to write this as $D = \sqrt{32^2 + 17^2 - 2(32)(17) \cos 51°}$.

   (b) What is the new bearing of the ship as measured from the port?
   The bearing is NθE, where $\theta = \arctan(x/y)$, with x and y given as in part (a). Equivalently, some used the law of sines to write $\theta = 180° - 74° - \arcsin\left(\frac{17 \sin 51°}{D}\right)$, with D as previously calculated.

5. (6 points) The planet Jupiter is about 390.4 million miles from earth, when the two planets are closest. At this point, Jupiter subtends an angle of about 0.01303° in our field of view. (That's pretty small, but you can see the disk and even a few of the moons of Jupiter with decent binoculars.) Find the approximate diameter of the planet Jupiter. 51

   $d \approx (390.4 \times 10^6 \text{ miles})(.01203°)(\pi/180°)$

6. (4 points) Assuming $\sin\alpha = -2/3$ and $270° < \alpha < 360°$, give exact algebraic values for the following. 55

   (a) $\cos(2\alpha)$ $\quad = 1/9$

   (b) $\cos(\alpha/2)$ $\quad = -\sqrt{\frac{1 + \sqrt{5}/3}{2}} = -\sqrt{\frac{3 + \sqrt{5}}{6}}$

7. (4 points) The moon is about 240,000 miles from earth (on average) and rotates about the earth about once every 28 days. Neglecting the motions of our earth about the sun (and the solar system in the galaxy, etc.) and assuming the earth's orbit is circular, how fast is the moon travelling through space, in miles per hour? 59

   $$v = r\omega = (240,000 \text{ mi})\frac{2\pi[\text{rad}]}{28(24 \text{ hr})} = 5000\pi/7 \text{ mph}$$

8. (4 points) A triangle has angles 55°, 60° and 65°. Its longest side (which is opposite the largest angle, of course) measures 10.00 inches. How long is its shortest side? 63

   $$\frac{(10.00 \text{ in.}) \sin 55°}{\sin 65°}$$

9. (4 points) A triangle has sides 95.000 cm, 100.000 cm, 105.000 cm. Find the angle of the triangle that is intermediate in size between the other two (not the biggest, not the smallest), to the nearest tenth of a degree (or give it in calculator-ready form). 67

   $$\arccos\left(\frac{105^2 + 95^2 - 100^2}{2(95)(105)}\right)$$

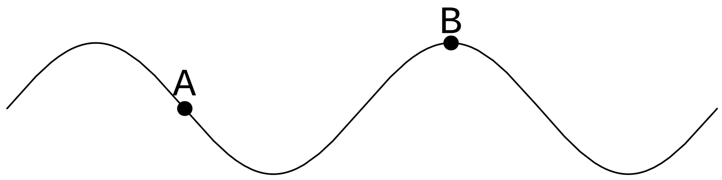
10. (12 points) Simplify each the following. Show all significant steps. 79

(a)
$$\frac{1-\tan^2 x}{1+\tan^2 x} \quad = \cos 2x$$

(b)
$$\sin(x+2\pi/3)+\sin(x+4\pi/3) \quad = -\sin x$$

(c)
$$\tan(2\arctan x) \quad = \frac{2x}{1-x^2}$$

11. (4 points) Find an equation that represents the following curve. The two points are $A = (\pi/3, 1)$ and $B = (5\pi/6, 3)$. 83



The most obvious forms of the answer are these:

$$y = 1 + 2\cos(3(x - 5\pi/6)),$$

$$y = 1 - 2\sin(3(x - \pi/3)),$$

$$y = 1 + 2\sin 3x$$

12. (4 points) Find an exact algebraic expression for $\tan 7\pi/12$. 87

Correct answers include these:

$$-2-\sqrt{3} = \frac{1+\sqrt{3}}{1-\sqrt{3}} = -\sqrt{\frac{2+\sqrt{3}}{2-\sqrt{3}}}$$

13. (12 points) Evaluate the following exactly. 99

(a) $\arcsin(\sin(8\pi/7)) \quad = -\pi/7$

(b) $\arccos(\cos(8\pi/7)) \quad = 6\pi/7$

(c) $\arctan(\tan(8\pi/7)) \quad = \pi/7$

(d) $\arcsin(\cos(8\pi/7)) \quad = -5\pi/14$

14. (12 points) Evaluate the following exactly. 111

(a) $\cos(\arcsin(-5/6))$ $= \sqrt{11}/6$

(b) $\cos(\arccos(-5/6))$ $= -5/6$

(c) $\cos(2\arccos(-5/6))$ $= 7/18$

15. (6 points) Find all real solutions. 117

$$\cos 2x = -\sin x$$

$x = \dfrac{\pi}{2} + 2n\pi, \dfrac{-\pi}{6} + 2n\pi,$ or $\dfrac{-5\pi}{6} + 2n\pi,$ where $n \in \mathbb{Z}$.

16. (6 points) Find all real solutions in $[0, 2\pi)$. 123

$$2\cos 3x + 1 = 0$$

$x \in \{2\pi/9, 4\pi/9, 8\pi/9, 10\pi/9, 14\pi/9, 16\pi/9\}$

17. (6 points) Find the exact (x, y)-coordinates of the point obtained by rotating the point $(-3, 1)$ counterclockwise through 30° about the origin. 129

$\left(\dfrac{-3\sqrt{3}-1}{2}, \dfrac{-3+\sqrt{3}}{2}\right)$

⋆⋆⋆ Extra Credit ⋆⋆⋆

(You may do these on the back of the previous page if you wish.)

A.) (⋆⋆) (This is a slight variation of problem #1.) When the morning sun has an angle of elevation of just 9.10°, how long is the shadow cast on a long hill having slope .215 by a tree standing exactly 15.3 feet tall?

With a calculator I got 41.71

B.) (⋆) Use Heron's formula to find the area of the triangle in problem #9.

$\sqrt{150(45)(50)(55)}$

C.) (⋆⋆) Circles of radius 4.00, 5.00 and 6.00 are mutually tangent. Find the area of the gap formed between them. (I'll draw a picture on the board if you wish).

I get $A = 9.7789$ when I compute it with a calculator.

D.) (⋆) Derive one of the product-to-sum identities.

E.) (⋆) Find the exact value of $\arccos(\cos(7/2))$.

$2\pi - \dfrac{7}{2}$

F.) (⋆) Find the exact value of $\arcsin(\sin(22))$.

$7\pi - 22$ (A wee bit tricky, this.)

G.) ($\star$) Give all exact solutions to the equation

$$\cos 2x = -2\cos x.$$

$x = \pm \arccos(\sqrt{3} - 1) + 2n\pi, \quad n \in \mathbb{Z}$

H.) $\star$ Find the exact (x, y)-coordinates of the point obtained by rotating the point $(-3, 1)$ counterclockwise through 30° about the point $(2, -1)$.

$\left(\dfrac{-5\sqrt{3}+2}{2}, \dfrac{1+2\sqrt{3}}{2}\right)$

I.) ($\star$) Write the complex number $\sqrt{3} - 2i$ in polar form.

$\sqrt{7}\ \mathrm{cis}(-\arctan \dfrac{2}{\sqrt{3}})$

J.) ($\star$) Write de Moivre's formula.

K.) ($\star\star$) Find the six 6^{th} roots of -1 and write them in the usual (rectangular) complex form.

Two are $\pm\, i$ and four more are $\dfrac{\pm\sqrt{3} \pm\ i}{2}$.

L.) ($\star \cdots \star$) Ask a question (or questions!) you wish I had asked and answer it. Points will certainly vary. (This is where you ask me the things you were ready for but that I didn't ask!)