Trigonometry	Fall 2008	Name:
Math 122-001	Final Exam	________________

The "No Calculators" Pages

Instructions: Finish these "No Calculators" pages before going to the rest. You may not return to this section once you turn on your calculator.

Put final answers in boxes when provided. Each st★rred problem is extra credit, and each ★ is worth 5 points.

The phrase *exact algebraic values* appears throughout the test. Quantities such as $\sqrt{3}$, $5/3$, etc., are exact algebraic values, as opposed to *numerical approximations*, such as 1.732, 1.666, etc., and *trigonometric expressions*, such as $\sin 60°$ and $\tan(5\pi/8)$, which are not.

N1.) (8 points) Find exact algebraic values for each of the following, where defined. Otherwise, write "undefined." 8

(a) $\tan 3\pi/4$ $\boxed{-1}$

(b) $\cos(-270°)$ $\boxed{0}$

(c) $\sin(5\pi/6)$ $\boxed{1/2}$

(d) $\csc(-\pi/2)$ $\boxed{-1}$

(e) $\sin 240°$ $\boxed{-\sqrt{3}/2}$

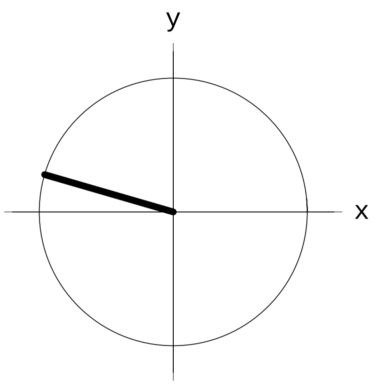
(f) $\cos 5\pi/4$ $\boxed{-\sqrt{2}/2}$

(g) $\cot 3\pi/2$ $\boxed{0}$

(h) $\sin 90°$ $\boxed{1}$

(★) Give the exact algebraic value for $\sin(1111111111111111111111111111111111111111111\pi/4)$ $\boxed{-\sqrt{2}/2}$

N2.) (4 points) An angle θ is shown below in standard position. Give an approximate numerical (decimal) value for $\sin\theta$, just by looking. No credit if you are off by more than two tenths! Zero credit for incorrect sign! Better guesses get better scores. 12



$\boxed{\approx .3}$

N3.) (8 points) Spew forth formulas. Write the standard identities for the following. 20

(a) $\cos(x+y)$ You can look most of these up.

(b) $\sin(a-b)$

(c) $\tan(\alpha-\beta)$

(d) $\sin 2t$

(e) $\cos 2\theta$ (write all three!)

(f) $\sin(B/2)$

(g) $\tan(\theta/2)$ (write at least two)

(h) $\sin(\pi+x)$ $= \quad -\sin x$

N4.) (3 points) Write the *exact algebraic values* of the functions $\sin\theta$, $\cos\theta$ and $\tan\theta$ for the angle θ in standard position having the point $(-\sqrt{3}, 2)$ on its terminal side. 23

$\sin\theta =$ $\boxed{2/\sqrt{7}}$

$\cos\theta =$ $\boxed{-\sqrt{3/7}}$

$\tan\theta =$ $\boxed{-2/\sqrt{3}}$

N5.) (6 points) Write 29

(a) a reciprocal identity involving $\cos\theta$, $\boxed{\frac{1}{\cos\theta} = \sec\theta \text{ or } \frac{1}{\sec\theta} = \cos\theta}$

(b) a Pythagorean identity involving $\cot\theta$, $\boxed{1 + \cot^2\theta = \csc^2\theta \text{ (and a few other forms)}}$

(c) a cofunction identity involving $\csc\theta$. $\boxed{\csc(90^\circ - \theta) = \sec\theta \text{ or } \sec(90^\circ - \theta) = \csc\theta}$

N6.) (6 points) Write the exact values of 35

(a) $\arctan(-1)$ $\boxed{-\pi/4}$

(b) $\arccos(-1)$ $\boxed{\pi}$

(c) $\arcsin(-1)$ $\boxed{-\pi/2}$

There were many errors here, mostly by using coterminal forms of the answers for the arctan and arcsin. For example, $\arctan(-1) = 3\pi/4$ and $\arcsin(-1) = 3\pi/2$ are false. Answers using degrees were fine.

N7.) (5 points) Verify the identity $\dfrac{\sin x - \cos x}{\cos x} + \dfrac{\sin x + \cos x}{\sin x} = \csc x \sec x$. Show all significant steps. 40

$$\begin{aligned}
\frac{\sin x - \cos x}{\cos x} + \frac{\sin x + \cos x}{\sin x} &= \frac{\sin x}{\cos x} - 1 + 1 + \frac{\cos x}{\sin x} \\
&= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \\
&= \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} \\
&= \frac{1}{\sin x \cos x} \\
&= \frac{1}{\sin x}\frac{1}{\cos x} \\
&= \csc x \sec x
\end{aligned}$$

N8.) (8 points) Simplify each the following. Show all significant steps. 48

(a) $\dfrac{1 - \cos 2x}{\sin 2x}$

The slick way, using a half-angle formula for tangent:

$$\begin{aligned}
\frac{1 - \cos 2x}{\sin 2x} &= \tan\left(\frac{2x}{2}\right) \\
&= \tan x.
\end{aligned}$$

Or the straightforward, longer way, using double-angle formulas for sine and cosine:

$$\begin{aligned}
\frac{1 - \cos 2x}{\sin 2x} &= \frac{1 - (1 - 2\sin^2 x)}{2 \sin x \cos x} \\
&= \frac{2\sin^2 x}{2 \sin x \cos x} \\
&= \frac{\sin x}{\cos x} \\
&= \tan x.
\end{aligned}$$

(b) $\tan(2 \arctan x)$

Use the double-angle formula for tangent:

$$\begin{aligned}
\tan(2 \arctan x) &= \frac{2 \tan(\arctan x)}{1 - \tan^2(\arctan x)} \\
&= \frac{2x}{1 - x^2}
\end{aligned}$$

N9.) (5 points) Find an exact algebraic expression for $\cos 7\pi/12$. (Correct answers have square roots but no trig functions, no angles, no π's.) 53

Since $7\pi/12 = \frac{1}{2}(7\pi/6)$, we can use the half-angle formula for cosine to get

$$\begin{aligned} \cos 7\pi/12 &= \cos\left(\frac{1}{2}(7\pi/6)\right) = \pm\sqrt{\frac{1+\cos(7\pi/6)}{2}} \\ &= \pm\sqrt{\frac{1+(-\sqrt{3}/2)}{2}} = \pm\sqrt{\frac{2-\sqrt{3}}{4}} \\ &= \pm\frac{\sqrt{2-\sqrt{3}}}{2}. \end{aligned}$$

Since $7\pi/12 \in$ QII and the cosine is negative there, our answer is $\boxed{-\frac{\sqrt{2-\sqrt{3}}}{2}}$.

Or we can use the fact that $7\pi/12 = 3\pi/12 + 4\pi/12 = \pi/4 + \pi/3$ to give

$$\begin{aligned} \cos 7\pi/12 &= \cos(\pi/4+\pi/3) = \cos(\pi/4)\cos(\pi/3) - \sin(\pi/4)\sin(\pi/3) \\ &= \frac{\sqrt{2}}{2}\cdot\frac{1}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} = \boxed{\frac{\sqrt{2}-\sqrt{6}}{4}}. \end{aligned}$$

N10.) (8 points) Evaluate the following exactly. 61

(a) $\arcsin(\sin(6\pi/7))$ $\boxed{\pi/7}$

(b) $\arccos(\cos(6\pi/7))$ $\boxed{6\pi/7}$

(c) $\arctan(\tan(6\pi/7))$ $\boxed{-\pi/7}$

(d) $\arcsin(\cos(6\pi/7))$ $\boxed{-5\pi/14}$

(Use the fact that $\cos(6\pi/7) = \sin(\pi/2 - 6\pi/7)$.)

($\star$) Give an approximate numerical value for $\arccos(-1/10)$ in decimal degrees. Better guesses get better scores.

A good guess is about 95°, just by drawing a decent picture. Wait, let me try to do better: I'll say about 96° for the following reason. Since $\arccos(0) = 90°$ and $\arccos(-1/2) = 120°$ are values on each side of and in the vicinity of our angle, and since the arc from 90° to 120° can be crudely approximated as a line (since it is relatively short), I'll use an easy linear interpolation. The number $-1/10$ is one-fifth the way from 0 to $-1/2$, so my guess will be the angle one-fifth the way from 90° to 120°. One-fifth of 30° is 6°, so I add 6° to 90°. Check with a calculator: the actual value is about 95.739°.

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The "Yes Calculators" Pages

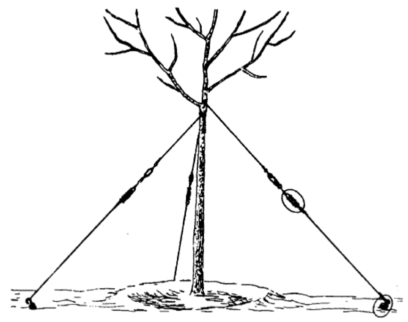
Instructions: Answer all problems correctly. Calculators are allowed on this part but *they must not be used to retrieve information or formulas.*

Put final answers in boxes when provided. Each st⋆rred problem is extra credit, and each ⋆ is worth 5 points.

The phrase *exact algebraic values* appears throughout the test. Quantities such as $\sqrt{3}$, $5/3$, etc., are exact algebraic values, as opposed to *numerical approximations*, such as 1.732, 1.666, etc., and *trigonometric expressions*, such as $\sin 60°$ and $\tan(5\pi/8)$, which are not.

1. (5 points) A transplanted tree is made to stand up straight by running taught wires ("guy wires") from anchors in the ground to points on the tree. Suppose that one of the wires meets the ground at an angle of 47° at a point 8.5 feet from the base of the tree. How long is the wire? 66

$8.5\,\text{ft} / \cos 47° \approx 12.5\,\text{ft}$



Guying large trees provides stability until roots grow into the soil.

From R.A. Harris, A.T. Leiser, and W.B. Davis, 1976, *Staking landscape trees*. Univ. Calif. Agr. Ext. Leaflet 2576.

2. (5 points) A tourist on the ground views the Eiffel tower, which has a height of 986.0 feet. The top of the tower has an angle of elevation of 24.36°. The tourist walks directly toward the tower until the distance to the base of the tower is exactly half what it was previously. What is the new angle of elevation to the top of the tower? 71

Let θ be the angle we want. We do not know the distance from the tourist to the base of the tower, so call that x. Draw the picture and you'll see two right triangles. The "outer" triangle informs us that $\tan 24.36° = \dfrac{986.0\,\text{ft}}{x}$, while the "inner" triangle tells us that

$$\tan\theta = \frac{986.0\,\text{ft}}{x/2} = 2 \cdot \frac{986.0\,\text{ft}}{x} = 2\,\tan 24.36°.$$

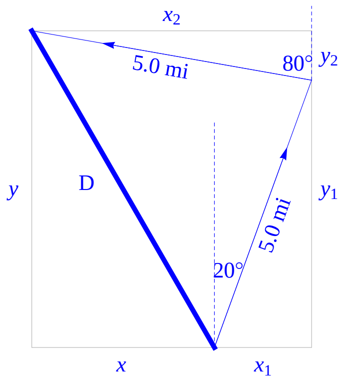
This means that $\theta = \arctan(2\,\tan 24.36°) \approx 42.16°$. (Notice that the value of the angle has nothing to do with the actual height of the tower—the entire problem scales to any height.)

3. (6 points) A ship leaves port and travels for 5.0 miles with a bearing of N20.0°E. The ship then turns (on a dime) and travels 5.0 more miles with a bearing of N80.0°W. How far is the ship from port? 77

Letting D denote the distance we want, see the "bounding box" below and confirm the following.

$$x_1 = 5.0\,\text{mi}\,\sin 20°, \quad \text{y}_1 = 5.0\,\text{mi}\,\cos 20°, \quad \text{x}_2 = 5.0\,\text{mi}\,\sin 80°, \quad \text{y}_1 = 5.0\,\text{mi}\,\cos 80°$$

$$x = x_2 - x_1, \quad y = y_1 + y_2, \quad D = \sqrt{x^2 + y^2}.$$



These give

$$x_1 \approx 1.7\,\text{mi}, \quad \text{y}_1 \approx 4.7\,\text{mi}, \quad \text{x}_2 \approx 4.9\,\text{mi}, \quad \text{y}_2 \approx 0.87\,\text{mi},$$

$$x \approx 3.2\,\text{mi}, \quad \text{y} \approx 5.6\,\text{mi}, \quad \text{D} \approx \boxed{6.4\,\text{mi}}.$$

You can also take a better approach now that we know the Law of Cosines. Notice that the angle δ across from D is (coincidentally) also 80°. Then the Law of Cosines gives

$$D = \sqrt{(5.0\,\text{mi})^2 + (5.0\,\text{mi})^2 - 2(5.0\,\text{mi})(5.0\,\text{mi})\cos 80°} \approx \boxed{6.4\,\text{mi}}.$$

Finally, in this particular example, the Law of Sines can actually be used, since the triangle is isosceles. The angles across from the two equal sides are each 50°, so

$$D = (5.00\,\text{mi})\sin 80° / \sin 50° \approx \boxed{6.4\,\text{mi}}.$$

4. (5 points) The Andromeda galaxy, believed to be disk-shaped, subtends about 3 degrees field of view in the sky. (Think of six full moons touching side by side. That's huge! The galaxy is so faint that we don't see its extent without powerful telescopes, but still...) The most recent estimates of the distance to the galaxy put it about 2.9 million light-years away. What is the approximate diameter of the galaxy in light-years? 82

The approximation using arc length ($D \approx R\theta$, with θ in radians) gives the diameter of the galaxy to be about

$$\approx (2.9 \times 10^6 \,\text{ly}) \times \left(3^\circ \frac{\pi}{180^\circ}\right) \approx \boxed{150{,}000\,\text{ly}}.$$

Sines and tangents can also be used, but the version above is (supposed to be!) easier for small angles when only an approximation is needed.

5. (6 points) Assuming $\cos\alpha = -2/3$ and $180^\circ < \alpha < 360^\circ$, give exact algebraic values for the following. 88

(a) $\cos(2\alpha)$

$\cos 2\alpha = 2\cos^2\alpha - 1 = 2(-2/3)^2 - 1 = \boxed{-1/9}$.

(b) $\sin(\alpha/2)$

Since $\sin(\alpha/2) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$, we are finished except to determine the sign. But $180^\circ < \alpha < 360^\circ$ implies that $90^\circ < \alpha/2 < 180^\circ$, so $\sin(\alpha/2) > 0$. Hence,

$$\sin(\alpha/2) = (+)\sqrt{\frac{1-(-2/3)}{2}} = \boxed{\sqrt{5/6}}.$$

6. (5 points) How fast is the tip of the minute hand of a clock moving (in cm/sec) if the minute hand is 18.00 cm long? 93

The angular speed of the minute hand is 1 revolution per hour. Converting to radians and seconds gives

$$v = r\omega = (18.00\,\text{cm})\left(\frac{2\pi}{3600\,\text{sec}}\right) \approx \boxed{.03142\frac{\text{cm}}{\text{sec}}}.$$

7. (5 points) A triangle has angles 10.0°, 20.0° and 150.0°. Its longest side (which is opposite the largest angle, of course) measures 10.00 inches. How long is its shortest side? 98

Let x denote this shortest side, which is across from the smallest (10.0°) angle. With our "ASA" info we can use the Law of Sines to get

$$x = \frac{10.00\,\text{in}\ \sin 10.0^\circ}{\sin 150.0^\circ} \approx \boxed{3.47\,\text{in}}.$$

8. (5 points) A triangle has sides 99.000 cm, 100.000 cm, 101.000 cm. Find the smallest angle of the triangle to the nearest hundredth of a degree (or give it in calculator-ready form). 103

Use the Law of Cosines for such "SSS" info. The smallest angle is across from the smallest side. Letting that be side a, we want angle A, where

$$a^2 = b^2 + c^2 - 2bc\,\cos A.$$

Therefore, with $a = 99.000\,\text{cm}$, $b = 100.000\,\text{cm}$, and $c = 101.000\,\text{cm}$, we get

$$A = \arccos\left(\frac{b^2 + c^2 - a^2}{2bc}\right) \approx \boxed{59.0125^\circ}.$$

9. (6 points) Find all real solutions in $[0, 2\pi)$. Show all significant steps. 109

$$2\cos^2 x = \cos x$$

We get

$$0 = 2\cos^2 x - \cos x = \cos x(2\cos x - 1),$$

so either $\cos x = 0$ or $\cos x = 1/2$. This implies that $x \in \{\boxed{\pi/2,\ 3\pi/2,\ \pi/3,\ 5\pi/3}\}$.

10. (6 points) Find the exact (x, y)-coordinates of the point obtained by rotating the point $(1, 2)$ counterclockwise through $60°$ about the origin. 115

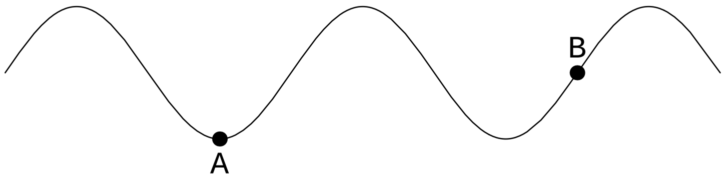
You need to know (or derive) the formulas,

$$x' = x\cos\theta - y\sin\theta, \quad y' = x\sin\theta + y\cos\theta.$$

Just plug in $x = 1,\ y = 2,\ \theta = 60°$ to get the rotated point,

$$(x', y') = \left(\frac{1}{2} - \sqrt{3}, \frac{\sqrt{3}}{2} + 1\right).$$

11. (5 points) Find an *exact* equation that represents the sinusoidal curve pictured below. The two points are $A = (0, -1)$ and $B = (5\pi/6, 7)$. Be sure to write (separately) the values for the period and amplitude, each worth one point partial credit. 120



The two points are separated by a horizontal distance of one-and-one-quarter periods, so we have $\frac{5}{4}P = \frac{5\pi}{6} - 0$, which gives the period, $\boxed{P = \frac{4}{5} \cdot \frac{5\pi}{6} = \frac{2\pi}{3}}$. This makes our "$b$" coefficient equal to 3 (by the relation $b = 2\pi/P$). The point B reveals the vertical shift 7. Looking between B and A gives the $\boxed{\text{amplitude } 7 - (-1) = 8}$. Staring at point B, I see a shifted $+\sin$ curve with horizontal offset $5\pi/6$. Therefore, the curve can be written $\boxed{y = 7 + 8\sin(3(x - 5\pi/6)).}$ Or, if you stare at point A and see a $-\cos$ curve with offset 0, then you may write, $\boxed{y = 7 - 8\cos(3x).}$

⋆⋆⋆ Extra Credit ⋆⋆⋆

(You may do these on the back of the previous page if you wish.)

A.) (⋆) When the morning sun has an angle of elevation of just 20.00°, a stiff and straight stalk of beans leans directly toward the sun, making an angle of 10.00° away from the vertical. The stalk is 4.00 feet in length. How long is its shadow?

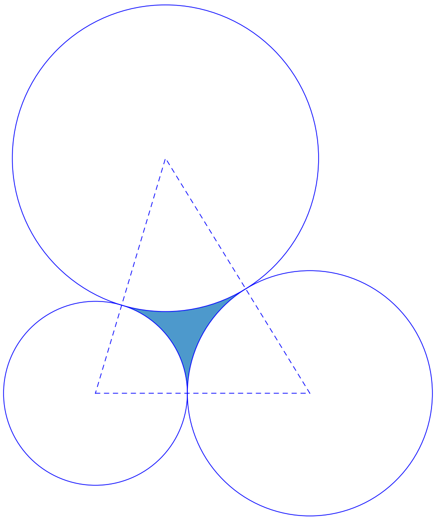
The Law of Sines gives the shadow's length as $\frac{4.00\,\text{ft}\ \sin 60.00°}{\sin 20.00°} \approx 10.13\,\text{ft}.$

B.) ($\star$) Use Heron's formula to find the area of the triangle in problem #8.

Just look up the formula and plug the stuff in. It's about $4329.26\,\text{cm}^2$

C.) ($\star\star$) Circles of radii 3.00, 4.00 and 5.00 inches are mutually tangent. Find the area of the gap formed between them. (I'll draw a picture on the board if you wish).

Here's the figure; you can work on it over the break! The numerical answer is about $2.3989\,\text{in}^2$.



D.) ($\star$) *Derive* the Law of Cosines formula.

From class notes or your text. (Italics on "derive" didn't help too much—most of you simply wrote the Law of Cosines in some form.)

E.) ($\star$) Find the exact value of $\arcsin(\sin(7/2))$.

Notice that this is $7/2$ **radians**. That's 3.5 radians, just past π, so it's in QIII. The sine is found by going horizontally from the terminal point of the angle to the y-axis, then continuing on to the "happy interval" of sine. The result is the angle in QIV that is $7/2-\pi$ radians below the x-axis, namely, the angle $-(7/2-\pi) = \pi - 7/2$. Check it on a calculator.

F.) ($\star$) Find the exact value of $\arccos(\cos(30))$. (Read this very carefully.)

The careful reading is to notice that it is *not* $30°$, it is 30 radians. This is trickier, but you can do some crude calculations:

$$30\,\text{rad} = \frac{30}{2\pi}\,\text{rev} = \frac{15}{\pi}\,\text{rev} \approx \frac{15}{3.14} \approx 4.77\,\text{rev},$$

so it coterminal with about $.77\,\text{rev}$, in QIV, almost straight down from the origin. If you subtract 5 revolutions, you get an angle in $(-\pi/2, 0)$. So

$$\arccos(\cos(30)) = \arccos(\cos(30 - 5(2\pi))) = 10\pi - 30.$$

G.) ($\star$) Give all exact solutions to the equation

$$6\cos^2 x - 7\cos x = 3$$

Writing $6\cos^2 x - 7\cos x - 3 = 0$, the quadratic formula gives

$$\cos x = \frac{7 \pm \sqrt{7^2 - 4(6)(-3)}}{2(6)} = \frac{7 \pm 11}{12}.$$

(Good grief, it actually factors—what are the odds??) This implies either $\cos x = 18/12 = 3/2$ or $\cos x = -1/3$. The first of these is impossible, since it would mean $\cos x > 1$. So $\cos x = -1/3$ is the only possibility. This means that $x = \arccos(-1/3)$ and $x = 2\pi - \arccos(-1/3)$ are the only two solutions in $[0, 2\pi)$. To get all solutions, add multiples of 2π to each. The set of all solutions is therefore

$$\{\arccos(-1/3) + 2\pi n : n \in \mathbb{Z}\} \cup \{-\arccos(-1/3) + 2\pi n : n \in \mathbb{Z}\},$$

where $\mathbb{Z}$ denotes the set of all integers. (You didn't have to write the result in such a snazzy way.)

H.) Find the exact (x, y)-coordinates of the point obtained by rotating the point $(1, 2)$ counterclockwise through $60°$ about the point $(2, -1)$.

The trick here is to use the formula in problem #10. But how? That formula rotates points about the origin only. But what you can do is this: (1) translate (shift) the pair of points so that the point of rotation becomes the origin; (2)do the rotation using the formula already mentioned; (3) shift back to the original position. Like so: You want to rotate $(1, 2)$ about $(2, -1)$, so move everything left by 2 and up by 1. This gives the point $(-1, 3)$ being rotated about the origin (still by $60°$). The result of this rotation is the point

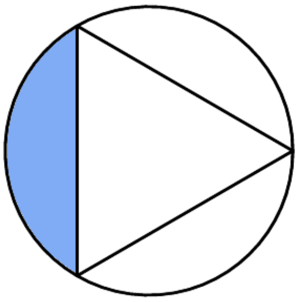
$$\left(-1/2 - 3\sqrt{3}/2,\ -\sqrt{3}/2 + 3/2\right).$$

Now just move that back—right by 2, down by 1. The result is

$$\left(3/2 - 3\sqrt{3}/2, -\sqrt{3}/2 + 1/2\right).$$

Check it using graph paper, a ruler, and a protractor, just as you've been doing all semester.

I.) ⋆ Consider the triangle inscribed in the circle below. If each side of the triangle has length equal to 1 unit, what is the area of the shaded region?



One approach among many: First show that the radius of the circle is $r = \sqrt{1/3}$. Show that the area of any equilateral triangle is $x^2\sqrt{3}/4$, where x is the length of its side. So our triangle has area $K = \sqrt{3}/4$. The shaded area is then

$$\frac{1}{3}(\pi r^2 - K) = \frac{1}{3}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right).$$

J.) (⋆· · · ⋆) Ask a question (or questions!) you wish I had asked and answer it (or them). Points will certainly vary. (Don't give repeats of what I've already asked—this is where you ask me the things you were ready for but that I didn't ask!)

My favorite response:
Q: *"Would you like some Christmas bonus points?"*
A: *"Why, yes, of course we would!"*
Well, I didn't give any such bonus for that (or anything else), but I did like the question!