Calc I — Fall 2008

MATH 221 — Name: ____________________

# Final Exam

**Instructions:** You may not use calculators or any electronic devices or notes of any kind.

1. (4 points) A ball is thrown upward with an initial speed of 64 feet per second. This implies (as you've proven, assuming that the acceleration due to gravity is about 32 ft/sec/sec) that the ball's height $h$ above the point of release is given approximately by the formula $h = -16t^2 + 64t$, where $h$ is given in feet and $t$ is given in seconds. 4

   (a) At what time $t_{\max}$ is the height of the ball at its maximum, and what is the maximum height?

   (b) Calculate the average velocity over the interval $0 \le t \le t_{\max}$.

2. (2 points) Write the precise "$\varepsilon$–$\delta$" definition of the limit. That is, write the precise statement that means $\lim_{x\to a} f(x) = L$. 6

3. (2 points) Write the precise definition of one of the following. 8

   (a) the infinite limit: $\lim_{x\to a} f(x) = \infty$.

   (b) the limit at infinity: $\lim_{x\to\infty} f(x) = L$.

4. (10 points) *Do any 5.* (One point extra for each additional correct limit.) Evaluate the following limits if they exist. Here and elsewhere, if the indicated limit does not exist, indicate if it goes to $\infty$, or $-\infty$, or if neither of these, simply write "D.N.E." (No partial credit; just write and circle the answers. Some are harder, some you can "stare down".) 18

(a) $\displaystyle \lim_{x\to 3^+} \frac{\sqrt{x+6}-3}{x-3}$

(b) $\displaystyle \lim_{x\to\infty} \frac{7x^2+x-1}{(2x+1)^3}$

(c) $\displaystyle \lim_{x\to\infty} \frac{\sqrt{9x^2-1}+x}{x-3}$

(d) $\displaystyle \lim_{x\to(\pi/2)^-} \tan x$

(e) $\displaystyle \lim_{x\to 0} \sin\frac{1}{x}$

(f) $\displaystyle \lim_{x\to 0} x\sin\frac{1}{x}$

(g) $\displaystyle \lim_{x\to 0} \frac{1}{x}\sin\frac{1}{x}$

(h) $\displaystyle \lim_{x\to\infty} x\sin\frac{1}{x}$

(i) $\displaystyle \lim_{x\to\infty} \cos\frac{1}{x}$

(j) $\displaystyle \lim_{x\to 2^+} \frac{\ln(x-2)}{x-2}$

5. (10 points) Write the standard formulas, and a bit more. Assume all functions are differentiable. 28

(a) the product rule for derivatives: write the formula for expanding $\frac{d}{dx}(f(x)g(x))$.

(b) extend the product rule to expand $\frac{d}{dx}(f(x)g(x)h(x))$.

(c) the quotient rule for derivatives: write the formula for expanding $\frac{d}{dx}\frac{f(x)}{g(x)}$.

(d) the chain rule for derivatives: write the formula for expanding $\frac{d}{dx}f(g(x))$.

(e) extend the chain rule to expand $\frac{d}{dx}f(g(h(x)))$.

6. (12 points) Evaluate each derivative. 40

(a) $\frac{d}{dx}\cos(2x+1)$

(d) $\frac{d}{dx}\arctan(5x)$

(b) $\frac{d}{dx}(x^2+3x)^7$

(e) $\frac{d}{dx}e^{e^x}$

(c) $\frac{d}{dx}\ln(x^3+x^2+x+1)$

(f) $\frac{d}{dx}\sin(\cos(x^2))$

7. (4 points) Find all critical points of the function $f(x) = \dfrac{x^2}{3x^2 - 2x + 1}$. 44

8. (4 points) Use a (limit) definition of the derivative to evaluate $\dfrac{d}{dx}\left(\dfrac{1}{\sqrt{x}}\right)$. 48

9. (4 points) Use the quotient rule (or power rule), the trig identity for $\tan x$ in terms of $\sin x$ and $\cos x$ (you'd better know it), and the formulas for the derivatives of $\sin x$ and $\cos x$ to derive the formula for the derivative of $\tan x$. **52**

10. (4 points) Use implicit differentiation to find $y'$ (in terms of $x$ and $y$) if **56**

$$x^2y^2 - xy = 7y^3.$$

11. Do one of the following. Part (b) is worth 4 points more than part (a). (You may do the other problem for $\frac{\star}{2}$ extra credit.) You may leave your answers in "calculator-ready" form. (That is, you needn't simplify your answers, but they must be numeric. However, a simplified answer is worth 2 extra points.)

(a) (8 points) At noon, ship A is 30 km due west of ship B. Ship A then sails due south while ship B sails due *east*. A short time later, ship A has traveled 15 km and ship B has sailed 25 km, and at that moment, ship A is moving at 35 km/hr, and ship B is traveling 5 km/hr. How fast is the distance between the two ships changing at that instant? [Note that the speeds of the ships must change. Note also that this problem has changed from a similar one on T3 (one word is different). And the answer isn't as nice.] 64

(b) A ferris wheel has a radius of 100 feet. (That seems huge. Is that big for ferris wheel? Never mind.) It turns at a rate of one revolution per minute. (Is that fast? Will the people hurl? Don't answer such questions.) The entire wheel of the ferris wheel sits 10 feet off the ground. (The people will need a ladder to get on and off!) Let $0°$ and $180°$ correspond to the bottom and top of the wheel, respectively, and let $G$ be the point on the ground directly below the wheel. When the wheel is rotating, a hurler, I mean, a passenger is at the point $P$ corresponding to $30°$. How fast is the distance between $P$ and $G$ changing? (It isn't critical to use it, but the Law of Cosines makes the calculations easier.)

12. Consider the function $g(x) = (1 - x)x^{2/3}$.

(a) (3 points) Find all critical points of $g$. 67

(b) (3 points) Identify each critical point as a relative minimum, a relative maximum, or neither. 70

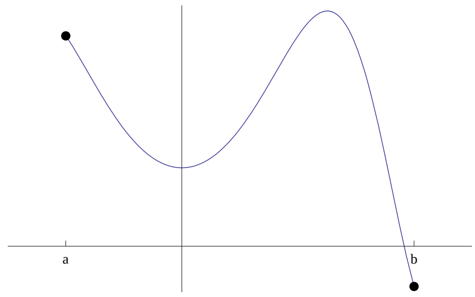
(c) (2 points) Find the absolute minimum and maximum values of $g$ on the interval $[-1, 2]$. 72

13. (2 points) State the Extreme Value Theorem (including all hypotheses and conclusions). 74

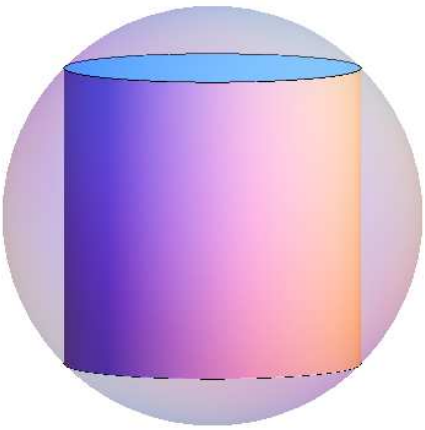
14. (2 points) State Rolle's Theorem (including all hypotheses and conclusions). 76

15. (2 points) State the theorem called the *Second Derivative Test.* 78

16. (2 points) Mark the graph below at the point(s) $(c, f(c))$ where $c$ satisfies the conclusions of the Mean Value Thorem on the interval $[a, b]$ shown. (Of course you're approximating this visually. Get as close as you can, so I'll know you understand.) 80




17. (6 points) Find the radius $r$ and the height $h$ of the right circular cylinder of greatest total surface area that can be inscribed in a sphere of radius $R$. Do not include top or bottom "caps" on the cylinder. (I sense panic. Okay, here's the formula for the surface area: $S = 2\pi rh$. It's just the circumference times the height.) 86



18. (3 points) Write the sum (using summation notation or not) for the approximation to the area under the curve $y = x^3$ on the interval $[1, 3]$ using 4 equal subintervals, and using the "midpoint rule" (meaning the sample points $x_i^*$ are the midpoints of the subintervals). You needn't actually calculate the sum! (Please don't unless you are lightning-fast and want to check your answer against an integral! See the next two problems.) 89

19. (3 points) Express the exact area under the curve $y = x^3$ on the interval $[1, 3]$ (from the previous problem) as the limit of a Riemann sum, i.e., using the definition of area. Use right-hand endpoints for the sample points $x_i^*$. Write everything explicitly so that a limit can be taken, but don't try to directly evaluate the limit. (However, that *would* be a good extra credit problem.) 92

20. (3 points) Without using a limit of a Riemann sum, calculate the exact area described in the previous two problems, namely, the area under the curve $y = x^3$ on the interval $[1, 3]$. 95

21. (4 points) State the Fundamental Theorem of Calculus, parts 1 and 2. 99

22. (12 points) *Do any four.* Do extras for 2 points each. Evaluate the following integrals (some definite, some indefinite). 111

(a) $\int (4x^4 - 2x^2)\, dx$

(b) $\int (2\sqrt{x} - 3\sqrt[3]{x} + \frac{1}{x^3})\, dx$

(c) $\int_0^{\pi} \sin\theta \, d\theta$

(d) $\int_0^1 10^t \, dt$

(e) $\int_2^5 (1/y)\, dy$

(f) $\int_0^2 f(x)\, dx$, where $f(x) = \begin{cases} x^4 & \text{if} \quad 0 \leq x < 1 \\ x^5 & \text{if} \quad 1 \leq x \leq 2 \end{cases}$

23. (4 points) Evaluate (simplify) the derivative of the function $\int_1^{x^2+x} \ln t \, dt$. 115

24. (6 points) A particle travels on the $x$-axis with a velocity given by $v(t) = 4 - 2t$ for $0 \leq t \leq 3$. 121
Notice that the velocity takes on both positive and negative values. (For 4 extra points, use the $v(t) = 4t - 2t^2$ instead.

(a) Find the net distance traveled by the particle.

(b) Find the total distance traveled by the particle.

Reminder: The *net* distance (also called the *displacement*) is the difference between the starting and ending positions, but you don't actually need to know what those positions are. The *total* distance is the length of "ground covered", as your car odometer might show.

25. (8 points) *Do any two.* Do any extras for 2 point seach. (These are from 5-3, with one tiny change.) 129

(a) $\int x^2 \sqrt{x}\, dx$

(b) $\int \frac{y + 5y^7}{y^3}\, dy$

(c) $\int_0^{\pi/4} \frac{1 + \cos^2 \theta}{\cos^2 \theta}\, d\theta$

(d) $\int_0^1 \frac{1}{4 + 4x^2}\, dx$

(e) $\int_{-1}^1 e^{u+1}\, du$

26. (6 points) *Do any two.* Two points each for extras. Use substitution to evaluate the integrals. 135

(a) $\displaystyle\int (3x-2)^{20}\,dx$

(b) $\displaystyle\int \frac{x}{(x^2+1)^2}\,dx$

(c) $\displaystyle\int \frac{e^x}{e^x+4}\,dx$

(d) $\displaystyle\int \frac{x}{\sqrt{1+2x}}\,dx$

## $\star \quad \star \quad \star$ Extras $\star \quad \star \quad \star$

Each st$\star$rred problem is extra credit and each $\star$ is worth 6 points. (These are just more problems, but harder. They're worth fewer points so that you're not unduly tempted.)

Feel free to do these on the back of the previous page or elsewhere. Just tell me where to look.

A. ($\star$) Prove, **using the $\varepsilon$–$\delta$ definition of the limit**, that $\lim_{x \to 3}(x^2 - x) = 6$.

B. ($\star$) Consider the function $g(x) = (1 - x)x^{2/3}$ given in problem #12, but this time on the interval $[a, b] = [-1, 1]$. Find all values of $c$ (if any) for which $\dfrac{g(b) - g(a)}{b - a} = g'(c)$. Discuss your result thoroughly in view of the MVT.

C. ($\star$) Again consider the function $g(x) = (1-x)x^{2/3}$ given in problem #12. Find all intervals of concavity of $g(x)$.

D. ($\star$) Evaluate the following limit. (Hint: Consider a definite integral.)

$$\lim_{N\to\infty} \frac{3}{N} \sum_{k=1}^{N} \frac{1}{2+3k/N}$$

E. ($\star$) Evaluate the integrals.

(a) $\displaystyle\int_0^3 \sqrt{9-x^2}\,dx$

(b) $\displaystyle\int_{-2}^{2} \left( x^2 + \frac{\sin x}{(x^4+x^2+1)^3} \right) dx$

(c) $\displaystyle\int_0^{6\pi} |\cos x|\,dx$

F. ($\star$) In problem #17 about the sphere and cylinder, is the optimal cylinder's volume greater or less than half the volume of the sphere? Justify thoroughly. (Your answer to problem #17 must be correct for any credit to be awarded here.)

G. ($\star \cdots \star$) Ask a question you wish I had asked and answer it. Points may vary. Few points (if any) awarded for "repeats" or trivial questions.