

T1

**Instructions:** You may NOT use calculators or any electronic devices or notes of any kind. Loads of points are possible on the test, but the highest grade that I will award is 115 points.

1. (6 points) A ball is thrown upward with an initial speed of 20 feet per second. This implies (if you believe Sir Isaac Newton) that its height  $h$  above the point of release is given approximately by the formula  $h = -16t^2 + 20t$ , where  $h$  is given in feet and  $t$  is given in seconds. We'll derive the formula later. In the meantime, you need only do this: calculate the average velocity over the interval  $1 \leq t \leq 2$ . 6

The average rate is  $-28$ .

$$\frac{\Delta h}{\Delta t} = \frac{h(t_2) - h(t_1)}{t_2 - t_1} = \frac{h(2) - h(1)}{2 - 1}.$$

Since  $h(2) = -16(2)^2 + 20(2) = -24$  and  $h(1) = -16(1)^2 + 20(1) = 4$ , we get

$$\frac{\Delta h}{\Delta t} = \frac{-24 - 4}{2 - 1} = -28.$$

2. (16 points) Evaluate the following limits if they exist. Here and elsewhere, if the indicated limit does not exist, indicate if it goes to  $\infty$ , or  $-\infty$ , or if neither of these, simply write "D.N.E." (No partial credit; just write and circle the answers.) 22

(a)  $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x + 1} = 0$

(e)  $\lim_{x \rightarrow (\pi/2)^+} \tan x = -\infty$

(b)  $\lim_{x \rightarrow 3} \frac{\sqrt{x+1} - 2}{x - 3} = 1/4$

(f)  $\lim_{x \rightarrow 0} \sin \frac{1}{x} = \text{D.N.E.}$

(c)  $\lim_{x \rightarrow 0} \frac{x - 1}{x^2(x + 2)} = -\infty$

(g)  $\lim_{x \rightarrow 0} x \cos \frac{1}{x} = 0$

(d)  $\lim_{x \rightarrow 1} \frac{2 - x}{(x - 1)^2} = \infty$

(h)  $\lim_{x \rightarrow 2^+} \ln(x - 2) = -\infty$

3. (8 points) Write the precise " $\varepsilon$ - $\delta$ " definition of the limit. That is, write the precise statement that means  $\lim_{x \rightarrow a} f(x) = L$ . 30

[See the text.]

4. (18 points) Evaluate the following limits if they exist. This time, you'll first break each limit down into its one-sided limits, then decide if the (two-sided) limit exists. (No partial credit; just write and circle the answers.)

(a) For  $f(x) = \frac{2}{x-3}$  and  $a = 3$ , evaluate the following.

i.  $\lim_{x \rightarrow a^+} f(x) = \infty$       ii.  $\lim_{x \rightarrow a^-} f(x) = -\infty$       iii.  $\lim_{x \rightarrow a} f(x) = \text{D.N.E.}$

(b) For  $f(x) = \frac{x^2 - 4}{x - 2}$  and  $a = 2$ , evaluate the following.

i.  $\lim_{x \rightarrow a^+} f(x) = 4$       ii.  $\lim_{x \rightarrow a^-} f(x) = 4$       iii.  $\lim_{x \rightarrow a} f(x) = 4$

(c) For  $f(x) = \csc x$  and  $a = \pi$ , evaluate the following.

i.  $\lim_{x \rightarrow a^+} f(x) = -\infty$       ii.  $\lim_{x \rightarrow a^-} f(x) = \infty$       iii.  $\lim_{x \rightarrow a} f(x) = \text{D.N.E.}$

(d) For  $f(x) = \frac{|x-1|}{x-1}$  and  $a = 1$ , evaluate the following.

i.  $\lim_{x \rightarrow a^+} f(x) = 1$       ii.  $\lim_{x \rightarrow a^-} f(x) = -1$       iii.  $\lim_{x \rightarrow a} f(x) = \text{D.N.E.}$

(e) For  $f(x) = \frac{x-1}{\sqrt{x}-1}$  and  $a = 1$ , evaluate the following.

i.  $\lim_{x \rightarrow a^+} f(x) = 2$       ii.  $\lim_{x \rightarrow a^-} f(x) = 2$       iii.  $\lim_{x \rightarrow a} f(x) = 2$

(f) For  $f(x) = \frac{x-1}{\sqrt{x}-1}$  and  $a = 1$ , evaluate the following.

i.  $\lim_{x \rightarrow a^+} f(x) = 0$       ii.  $\lim_{x \rightarrow a^-} f(x) = \text{D.N.E.}$       iii.  $\lim_{x \rightarrow a} f(x) = \text{D.N.E.}$

5. (4 points) Write the precise definition of the following infinite limit:  $\lim_{x \rightarrow a} f(x) = \infty$ .

[See the text.]

6. (4 points) Write the precise definition of the following limit at infinity:  $\lim_{x \rightarrow \infty} f(x) = L$ .

[See the text.]

7. (10 points) For the function whose graph is shown, list all values of  $c$ , if any, at which

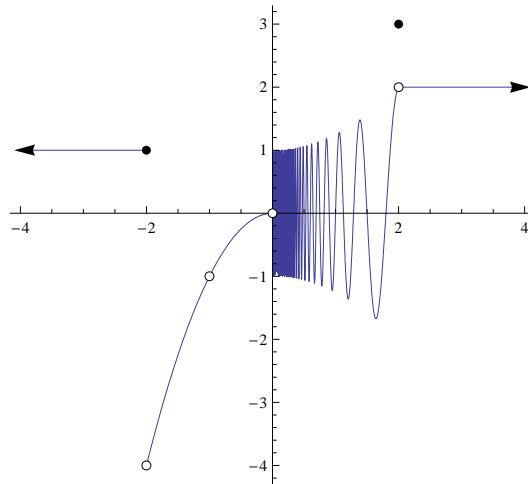
(a)  $\lim_{x \rightarrow c^+} f(x)$  does not exist, at  $c = 0$  only

(b)  $\lim_{x \rightarrow c^-} f(x)$  does not exist, none; the limit exists from the left at each  $c$

(c)  $\lim_{x \rightarrow c} f(x)$  does not exist, at  $c = 0, c = -2$

(d)  $f(x)$  is not continuous at  $x = c$ , at  $c = -2, -1, 0, 2$

(e)  $f(x)$  has a removable discontinuity at  $x = c$ . at  $c = -1, 2$



8. (4 points) Write the definition of “the function  $f(x)$  is continuous at  $x = a$ .”

[See the text.]

9. (10 points) For the function whose piecewise definition is given below, list all values of  $c$ , if any, at which 80

(a)  $\lim_{x \rightarrow c^+} f(x)$  does not exist,  $c = -3$

(b)  $\lim_{x \rightarrow c^-} f(x)$  does not exist,  $c = -3, 1$

(c)  $\lim_{x \rightarrow c} f(x)$  does not exist,  $c = -3, -1, 1$

(d)  $f(x)$  is not continuous at  $x = c$ ,  $c = -3, -1, 1$

(e)  $f(x)$  has a removable discontinuity at  $x = c$ . none

$$f(x) = \begin{cases} \frac{2}{x+3} & \text{if } x < -1 \\ -x + 1 & \text{if } -1 < x < 0 \\ \cos\left(\frac{2\pi}{x-1}\right) & \text{if } 0 \leq x < 1 \\ \frac{3}{x+2} & \text{if } x \geq 1 \end{cases}$$

10. (4 points) Write the precise statement and conclusion of the Intermediate Value Theorem. 84

[See the text.]

11. (6 points) Prove, using flawless logic and exposition, that the Intermediate Value Theorem implies that there exists a number  $x$  that is equal to exactly one more than its cube. 90

We're being asked to prove that there exists a number  $x$  for which  $x = x^3 + 1$ . Rewriting this as  $x^3 - x + 1 = 0$ , we are asking if the function  $f(x) = x^3 - x + 1$  is ever equal to 0. The hint of the IVT now suggests we (1) wonder if  $f(x)$  is continuous (it is) and (2) look for two values of  $x$  for which  $f(x)$  has opposite signs. An obvious fact is that  $f(0) = 1$ , so we're done if we can show  $f(x) < 0$  for some  $x$ . Noting that  $x^3$  is a relatively large negative number for  $x < -1$ , we look there. (In fact,  $\lim_{x \rightarrow -\infty} f(x) = -\infty$ .) We try  $x = -2$ . Sure enough,  $f(-2) = -5$ , and that does it. So here's a bare-bones proof:

Let  $f(x) = x^3 - x + 1$ . Then  $f$  is continuous, since  $f$  is a polynomial. Also,  $f(0) = 1$  and  $f(-2) = -5$ . Since  $f(-2) < 0 < f(0)$  and  $f$  is continuous, the Intermediate Value Theorem implies there exists a number  $c \in (-2, 0)$  for which  $f(c) = 0$ . Thus  $c^3 - c + 1 = 0$ , that is,  $c = c^3 + 1$ . □

We also might note that  $f(-1) = 1$ , so we can actually show there is a solution between  $-2$  and  $-1$ .

12. (15 points) Evaluate the following limits.

(a)  $\lim_{x \rightarrow -\infty} \frac{x+1}{2x^2+1} = 0$

(b)  $\lim_{x \rightarrow -\infty} \frac{2x^2+1}{x+1} = -\infty$

(c)  $\lim_{x \rightarrow \infty} \frac{7x^2+x-1}{(2x+1)^2} = 7/4$

(d)  $\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2+1}+x}{x-3} = 3$

(e)  $\lim_{x \rightarrow \infty} (\sqrt{x^2+3x}+2x) = \infty$

Too easy! Try the same one but let  $x \rightarrow -\infty$ .

★ ★ ★ Extras ★ ★ ★

Each starred problem is extra credit and each ★ is worth 5 points. (These are just more problems, but harder. They're worth fewer points so that you're not unduly tempted.)

Feel free to do these on the back of the previous page or elsewhere. Just tell me where to look.

A. (★) Evaluate  $\lim_{x \rightarrow 4} \frac{\sqrt{2x+1}-3}{\sqrt{x+12}-4} = 8/3$

B. (★)

(a) Evaluate  $\lim_{x \rightarrow 1} \frac{x^{17}-1}{x^{14}-1} = 17/14$

(b) Evaluate  $\lim_{x \rightarrow 2} \frac{x^5-32}{x^3-8} = 20/3$

C. (★) Use the Squeeze Theorem (and explain your reasoning) to decide the limit:  $\lim_{x \rightarrow 0} f(x)$ , where

$$f(x) = \begin{cases} x^3 & \text{if } x \text{ is rational} \\ -x^3 & \text{if } x \text{ is irrational.} \end{cases}$$

The limit is 0. To see this notice that whether  $x$  is rational or not,  $|f(x)| = |x^3|$ , so that

$$-|x^3| \leq f(x) \leq |x^3|$$

for all  $x$ . To apply the Squeeze Theorem, we note that both  $-|x^3|$  and  $|x^3|$  go to zero as  $x \rightarrow 0$ . Thus  $f(x)$  is “squeezed” to zero as  $x \rightarrow 0$ .

D. (★) Prove, **using the  $\varepsilon$ - $\delta$  definition of the limit**, that  $\lim_{x \rightarrow 4} (x^2 - x) = 12$ .

You're thinking my way if your proof starts out like so: “Supposing  $\varepsilon > 0$  is given, let  $\delta = \min(1, \varepsilon/8)$ . ...”

E. (★...★) Surely I forgot something you were ready for. Ask a question you wish I had asked and answer it. Points may vary. Offer void where prohibited by law.