Calc I
MATH 221-001

Fall 2008
Name: ______________

# T2 (with answers)

**Instructions:** Do all problems correctly. You may NOT use calculators or any electronic devices or notes of any kind. Loads of points are possible on the test, but the highest grade that I will award is 110 points.

1. (8 points) Data dump! Purge for credit! Write the indicated formulas. 8

   Look these up.

   (a) the product rule for derivatives

   (b) the quotient rule for derivatives

   (c) the chain rule for derivatives

   (d) the two equivalent limits that yield the derivative $f'(a)$

   $f'(a) = \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$.

2. (8 points) More dumping! Write the formulas for each derivative. 16

   Look these up.

   (a) $\dfrac{d}{dx}\cos x$

   (b) $\dfrac{d}{dx}x^n$

   (c) $\dfrac{d}{dx}\log x$

   This is from a section after 3.5, so you can wait until we get there. But since we did discuss derivatives of logs in class, here's some more info.

   As written, one can take $\log x$ to be synonymous with the "common logarithm", $\log_{10} x$. (But be aware that in upper level classes and mathematical writings, it is rare to use the common log and typically $\log x$ is taken synonymously with $\ln x = \log_e x$, the "natural log".) The answer is that

   $$\frac{d}{dx}\log_{10} x = \frac{\ln 10}{x},$$

   and here's why:

Writing $f(x) = \log_{10} x$ ("logarithmic form") is equivalent to writing $10^{f(x)} = x$ (the corresponding "exponential form"). Differentiating, we have

$$\frac{d}{dx}\,10^{f(x)} = \frac{d}{dx}\,x\,.$$

Since $\frac{d}{dx}a^x = (\ln a)\,a^x$, the chain rule tells us that $\frac{d}{dx}a^{f(x)} = (\ln a)\,a^{f(x)}f'(x)$, so the line above becomes

$$(\ln 10)\,10^{f(x)}f'(x) = 1.$$

Solving for $f'(x)$ (which is what we're after), we have $f'(x) =$

$$\frac{d}{dx}\log_{10} x = \frac{\ln 10}{10^{f(x)}} = \frac{\ln 10}{x},$$

the last equality following from the exponential form written above.

(d) $\dfrac{d}{dx}\tan x$

(e) $\dfrac{d}{dx}\arctan x$

Again, this is from a later section, so you can pass it by for now. But as I said in class, we can already handle inverses. Let $f(x) = \arctan x$; we want $f'(x)$. Follow the lines below to see that $f'(x) = \dfrac{1}{1+x^2}$.

$$\begin{aligned} f(x) = \arctan x \quad &\Rightarrow \quad \tan f(x) = x \\ &\Rightarrow \quad \frac{d}{dx}\tan f(x) = \frac{d}{dx}x = 1 \\ &\Rightarrow \quad \left(\sec^2 f(x)\right) f'(x) = 1 \\ &\Rightarrow \quad f'(x) = \frac{1}{\sec^2 f(x)}. \end{aligned}$$

Using a trig identity and simplifying,

$$\begin{aligned} f'(x) &= \frac{1}{1+\tan^2 f(x)} \\ &= \frac{1}{1+\tan^2 \arctan x} \\ &= \frac{1}{1+x^2}. \end{aligned}$$

(f) $\dfrac{d}{dx}e^x$

(g) $\dfrac{d}{dx}a^x$ (when $a$ is a constant)

(h) $\dfrac{d}{dx}\sec x$

3. (8 points) Find $f'(x)$ if $f(x) = \dfrac{x^2}{3x^4 - 2x + 1}$. 24

Use the quotient rule.

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\frac{x^2}{3x^4-2x+1} \\
&= \frac{(3x^4-2x+1)[x^2]' - [3x^4-2x+1]'(x^2)}{(3x^4-2x+1)^2} \\
&= \frac{(3x^4-2x+1)(2x) - (12x^3-2)(x^2)}{(3x^4-2x+1)^2} \qquad \text{(I should have said, "... and simplify.")} \\
&= \frac{(6x^5-4x^2+2x) - (12x^5-2x^2)}{(3x^4-2x+1)^2} \\
&= \frac{-6x^5-2x^2+2x}{(3x^4-2x+1)^2} \\
&= \frac{-2x(3x^4+x-1)}{(3x^4-2x+1)^2}
\end{aligned}
$$

4. (8 points) Find $y'$ if $y = \sqrt{3x-1}\,\ln(x^2+x)$. 32

This has a log, so it was intended for a subsequent section, but we do know how to do it. Rewrite the radical as a power and use the fact that $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$. The chain rule, power rule, and product rule will deliver it:

$$
\begin{aligned}
y' &= \left[(3x-1)^{1/2}\right]' \ln(x^2+x) + (3x-1)^{1/2}\left[\ln(x^2+x)\right]' \\
&= \left[\frac{1}{2}(3x-1)^{-1/2}(3)\right]\ln(x^2+x) + (3x-1)^{1/2}\left[\frac{2x+1}{x^2+x}\right] \\
&= \frac{3\ln(x^2+x)}{2\sqrt{3x-1}} + \frac{(2x+1)\sqrt{3x-1}}{x^2+x}.
\end{aligned}
$$

5. (8 points) Find $\dfrac{d}{dx}(3x-\cos(5x))^4$. 40

Let $f(x) = (3x-\cos(5x))^4$. Then

$$
\begin{aligned}
f'(x) &= 4\,(3x-\cos(5x))^3\,[3x-\cos(5x)]' \\
&= 4\,(3x-\cos(5x))^3\,(3+5\sin(5x))
\end{aligned}
$$

6. (8 points) Find $g'(t)$ when $g(t) = \dfrac{e^{-x^2}}{\pi^2 + 7}$. **48**

Okay, a bit of a trick/quiz was built in here to see if is noticed that the denominator is a constant. Unfortunately there is also a typo, giving mixed $x$'s and $t$'s! So let's go with the intended version,

$$g(t) = \frac{e^{-t^2}}{\pi^2 + 7}.$$

Then

$$\begin{aligned} g'(t) &= \frac{d}{dt}\frac{e^{-t^2}}{\pi^2+7} \\ &= \frac{1}{\pi^2+7}\frac{d}{dt}e^{-t^2} \\ &= \frac{1}{\pi^2+7}e^{-t^2}(-2t) \\ &= \frac{-2t\,e^{-t^2}}{\pi^2+7} \end{aligned}$$

7. (8 points) Use a (limit) definition of the derivative to evaluate $\dfrac{d}{dx}\left(\dfrac{1}{x^2}\right)$. **56**

Letting $f(x) = \frac{1}{x^2}$, we have,

$$\begin{aligned} f'(x) &= \lim_{t\to x}\frac{f(t)-f(x)}{t-x} \\ &= \lim_{t\to x}\frac{\frac{1}{t^2}-\frac{1}{x^2}}{t-x} \\ &= \lim_{t\to x}\frac{x^2-t^2}{t^2\,x^2\,(t-x)} \\ &= \lim_{t\to x}\frac{(x-t)(x+t)}{t^2\,x^2\,(t-x)} \\ &= \lim_{t\to x}\frac{-(x+t)}{t^2\,x^2} \\ &= \frac{-2x}{x^4} \\ &= -\frac{2}{x^3}. \end{aligned}$$

Notice that you can check your answer, since you know that $\dfrac{d}{dx}x^{-2} = -2x^{-3}$.

8. (8 points) Use the quotient rule (or power rule), the trig identity for $\sec x$ in terms of $\cos x$ (you'd better know it), and the formula you gave in problem #2a (let's hope it is correct) to derive the formula for the derivative of $\sec x$. (If you aren't sure what I'm asking for here, my advice is to skip this one.) 64

$$
\begin{aligned}
\frac{d}{dx}\sec x &= \frac{d}{dx}(\cos x)^{-1} \\
&= (-1)(\cos x)^{-2}\frac{d}{dx}\cos x \\
&= \frac{-1}{\cos^2 x}(-\sin x) \\
&= \frac{1}{\cos x}\,\frac{\sin}{\cos x} \\
&= \sec x\,\tan x
\end{aligned}
$$

9. (8 points) Evaluate $\dfrac{d}{dx}\ln\left(\dfrac{(2x-1)^7}{(3x^2+1)^5}\right)$. 72

From a later section, but we can do it. The trick is to first simplify using the properties of logs, then differentiate. Letting $y$ denote the function being differentiated, we have

$$
\begin{aligned}
y' &= \frac{d}{dx}\left[7\ln(2x-1) - 5\ln(3x^2+1)\right] \\
&= 7\left(\frac{2}{2x-1}\right) - 5\left(\frac{6x}{3x^2+1}\right) \\
&= \frac{14}{2x-1} - \frac{30x}{3x^2+1}
\end{aligned}
$$

10. (8 points) Evaluate $\dfrac{d}{dx}\left(x^2+1\right)^{1/x}$. 80

This uses a trick called "logarithmic differentiation" that is discussed in a later section. All it amounts to is taking the log of both sides, then differentiating. We could instead use the trick we have already seen in class.

$$
\begin{aligned}
\frac{d}{dx}\left(x^2+1\right)^{1/x} &= \frac{d}{dx}\left(e^{\ln\left(x^2+1\right)}\right)^{1/x} \\
&= \frac{d}{dx}\left(e^{(1/x)\ln\left(x^2+1\right)}\right) \\
&= \left(e^{(1/x)\ln\left(x^2+1\right)}\right)\frac{d}{dx}\left[\frac{1}{x}\ln\left(x^2+1\right)\right] \\
&= \left(e^{(1/x)\ln\left(x^2+1\right)}\right)\left[\frac{-1}{x^2}\ln\left(x^2+1\right) + \frac{1}{x}\left(\frac{2x}{x^2+1}\right)\right]
\end{aligned}
$$

the last part coming from the product rule, power rule and chain rule

$$
= \left(x^2+1\right)^{1/x}\left(\frac{2}{x^2+1} - \frac{1}{x^2}\ln\left(x^2+1\right)\right).
$$

11. (8 points) Use implicit differentiation to find $y'$ (in terms of $x$ and $y$) if 88

$$xy^2 - e^{3y} = 7x.$$

From a later section, but try it by letting $y = f(x)$, differentiating, and solving for $f'(x)$.

12. (8 points) Find an equation for the slope of the line tangent to the curve 96

$$x^3 + y^3 = 3x - y + 4$$

at the point $(2, 1)$.

Uses implicit differentiation from a later section.

13. (6 points) Evaluate $\lim_{t\to 0} \dfrac{\sin^2(3t)}{t\,\sin(5t)}$. 102

This one we can handle in several ways. Whatever method is used, you are relying on the fact that $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$. Here's one view: insert convenient factors to make your quantities nice.

$$\begin{aligned} \frac{\sin^2(3t)}{t\,\sin(5t)} &= \frac{9}{5}\left(\frac{\sin(3t)}{3t}\right)^2 \frac{5t}{\sin(5t)} \\ &\to \frac{9}{5}\cdot 1\cdot 1 \text{ as } t\to 0 \end{aligned}$$

Ultimately, however, I hope you learn to "stare it down" like so: for $\theta \approx 0$, we have $\sin\theta \approx \theta$. That means for $t \approx 0$, we have

$$\frac{\sin^2(3t)}{t\,\sin(5t)} \approx \frac{(3t)^2}{t(5t)} = \frac{9t^2}{5t^2} = 9/5.$$

Like all tricks, you need to use care, and there can be some subtleties with things like this. We'll talk more about such things later.

14. (8 points) Derive the formula for the derivative of $\arccos x$. 110

See #2(e) for the trick used here. The answer is $\dfrac{-1}{\sqrt{1-x^2}}$. Give it a try.

15. (8 points) State the theorem relating continuity to differentiability. (This was nicknamed the "Joe Biden Theorem" in class, for reasons I cannot recall. You may *prove* the theorem for extra credit — see problem E.) 118

You don't need to know the Biden gag from last year's class—the theorem relating continuity to differentiability is this: *If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.*

## $\star \quad \star \quad \star$ Extras $\star \quad \star \quad \star$

Each st$\star$rred problem is extra credit and each $\star$ is worth 5 points.
(Feel free to do these on the back of the previous page or elsewhere. Just tell me where to look.)

A. ($\star$) Find a formula, in terms of $x$ and the positive integer $n$, for the $n$'th derivative $f^{(n)}(x)$ of the function $f(x) = x\,e^x$. For a few extra points, use the function $f(x) = x\,e^{ax}$ instead, where $a$ is a constant.

B. ($\star\star$) Consider the curve $x^3 + y^3 = 3x - y + 4$ given in problem #12.

(a) Show that the curve has two horizontal tangents.

(b) Show that the curve has no vertical tangents.

(c) Find the exact $(x, y)$ coordinates for *one* of the horizontal tangents.

C. ($\star$) One-sided derivatives.

(a) Write the formal (limit) definition of $f'_+(x)$, the right-hand derivative of a function $f(x)$.

(b) The function $f(x) = |x^2 - x| + 2x$ fails to be differentiable at $x = 0$ and at $x = 1$, but the one-sided derivatives exist there. Calculate (using any method you wish) the left-hand and right-hand derivatives $f'_-(x)$ and $f'_+(x)$ at $x = 0$.

D. ($\star$) The *symmetric derivative* of a function $f(x)$ is defined to be

$$f'_s(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h},$$

provided the limit exists. Calculate $f'_s(0)$ for the function $f(x) = |x^2 - x| + 2x$.

E. ($\star$) Prove the Joe Biden Theorem. (See problem #15.)

F. ($\star$) Prove that a differentiability implies symmetric differentiability. That is, prove that if $f'(a)$ exists, then $f'_s(a)$ also exists.

G. ($\star$) Give an example of a function that is differentiable but whose derivative is not continuous.

H. ($\star$) Evaluate the limit by using the trick of converting to appropriate derivatives (perhaps after inserting a convenient form of the number 1). Show all work. No credit if you use theorems from later in the course.

$$\lim_{h \to 0} \frac{2^h - 1}{\cos 2h - 1}$$

I. ($\star \cdots \star$) Surely I forgot something you were ready for. Ask a question you wish I had asked and answer it. Points may vary. Offer void where prohibited by law.