Calc I
MATH 221-001

Fall 2008
Name: ____________

T3 (with answers)

Instructions: Do all problems correctly. You may NOT use calculators or any electronic devices or notes of any kind. Loads of points are possible on the test, but the highest grade that I will award is 115 points. Each ⋆ is extra credit and worth 5 points.

1. (20 points) Do one of the following. (You may do the other for one ⋆ extra credit.) You may leave your answers in "calculator-ready" form. (That is, you needn't simplify your answers, but they must be numeric. However, a simplified answers is worth 3 extra points.) 20

   (a) At noon, ship A is 30 km due west of ship B. Ship A then sails due south while ship B sails due north. A short time later, ship A has traveled 15 km and ship B has sailed 25 km, and at that moment, ship A is moving at 35 km/hr, and ship B is traveling 5 km/hr. How fast is the distance between the two ships changing at that instant? [Note: the speeds of the ships are not constant.]

   Let a and b be the distances sailed by ships A and B, respectively, since noon. Then their distance D apart (in km) is given by $D^2 = 30^2 + (a+b)^2$. Differentiate to get

   $$2D\frac{dD}{dt} = 2(a+b)\left(\frac{da}{dt} + \frac{db}{dt}\right),$$

   which is easily solved for dD/dt. When $a = 15$ and $b = 25$, you get $D = \sqrt{30^2 + 40^2} = 50$ (how convenient). Plug that, along with the instantaneous values given for da/dt and db/dt to get $dD/dt = 32$ km/hr.

   (b) A light on an ambulance rotates 3 times per second (about a vertical axis, like a lighthouse.). The ambulance's light throws a spot (of light) onto a long, straight wall that runs due north and south. The point P on the wall that is nearest the light is 20 meters from the light. How fast (in meters per second) is the spot moving along the wall when it is 10 meters from P?

   Let X be the point at the spot, O the point of the light's source, and let $\theta = \angle POX$. Let $x = PX$, the distance of the spot from P. Then $x = 20\tan\theta$. Diff it to get $dx/dt = 20(\sec^2\theta)(d\theta/dt)$. We are given that $d\theta/dt = 3$ rev/sec$= 6\pi$ rad/sec. When $x = 10$ we have $\sec^2\theta = 1 + \tan^2\theta = 1 + (10/20)^2 = 5/4$. Altogether, at this point we get $dx/dt = 20(5/4)(6\pi) = 150\pi$ m/sec.

2. (8 points) For $y = 1/(x^2+1)$, find the differential dy and evaluate dy with the values $x = 3$ and $dx = 1/10$ (and simplify). 28

   $dy = (dy/dx)dx = -(2x/(x^2+1)^2)\,dx$. Plug in the values to get $dy = -(6/100)(1/10) = -6/1000$.

(⋆) Use differentials to give a formula for the approximate relative error in the calculation of the volume of a sphere when the radius is r and the error in measuring r is Δr.

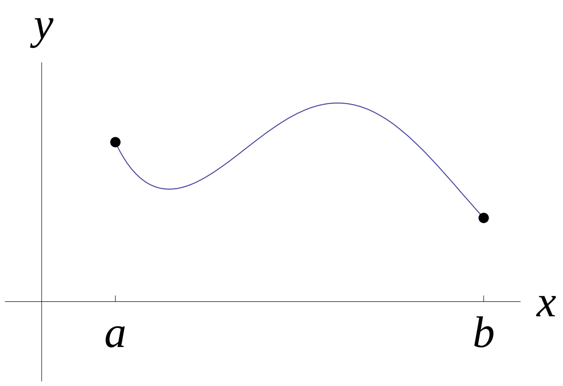
Since $V = 4\pi r^3/3$, we have $dV = 4\pi r^2 dr$. Relative error is defined to be dV/V, which here is $(4\pi r^2 \Delta r)/(4\pi r^3/3) = 3\Delta r/r$.

3. (12 points) Find the absolute maximum and minimum values of the function $q(x) = x^4 - 2x^2 + 3$ on the interval $[-2, 3]$. 40

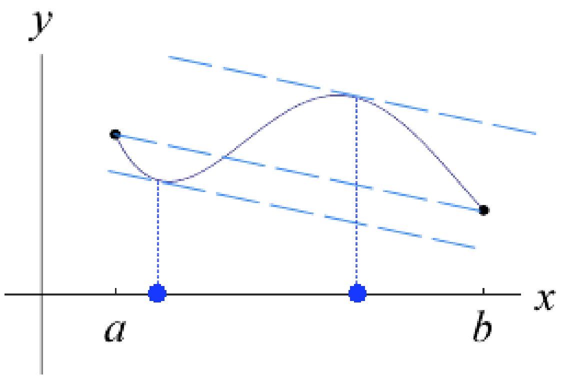
The critical points of q are found by solving $q'(x) = 0$, or $4x^3 - 4x = 0$. Factor and solve to get $x \in \{-1, 0, 1\}$. These are all in the interval $[-2, 3]$ (too bad!) and we have $q(-1) = q(1) = 2$, $q(0) = 3$. Check the endpoints, too: $q(-2) = 11$, $q(3) = 66$. So the max of $q(x)$ on $[-2, 3]$ is 66, the minimum is 2.

4. (5 points) State the Mean Value Theorem (including all hypotheses and conclusions). 45

5. (6 points) Mark the graph below at the point(s) $(c, f(c))$ where c satisfies the conclusions of the Mean Value Thorem on the interval $[a, b]$. (Of course you're approximating this visually. Get as close as you can, so I'll know you understand.) 51



The answer consists of the two points marked in blue, below. The other markings are a guide. Note: these points do NOT correspond to the local min and max, which are very close by.



6. (10 points) For the function $f(x) = x^3 + x$ on the interval $[-2, 0]$, find all values of c (if any) satisfied by the conclusion of the Mean Value Theorem. 61

Solve the following for c: $f'(c) = \dfrac{f(0) - f(-2)}{0 - (-2)}$. That is, solve: $3c^2 + 1 = 5$. You get $c = \pm 2/\sqrt{3}$. Of these, only $-2/\sqrt{3}$ is in $(-2, 0)$, and so $c = -2/\sqrt{3}$ is the only solution.

7. (6 points) Give the definition of a *local maximum* of a function. 67

8. (6 points) State the theorem called the *First Derivative Test*. 73

9. (6 points) State the theorem called the *Increasing/Decreasing Test*. (It should be called the *Monotonicity Test*, if you ask me.) 79

10. (6 points) Give the definition of *concave upward*. 85

11. (6 points) State the theorem called the *Concavity Test*. 91

12. (20 points) Choose (and circle) one of the two functions below for this problem. 111

   (i) $g(x) = 3x^{2/3} - 2x$

   (ii) $h(x) = (x-2)^3(x+1)$

   For the selected function, do the following.

   (a) Find the intervals of increase or decrease.

   (b) Find the local max and min values (if any).

   (c) Find the intervals of concavity and inflection points (if any).

   (d) Sketch the graph. (Make your own x- and y-axes, but be neat.)

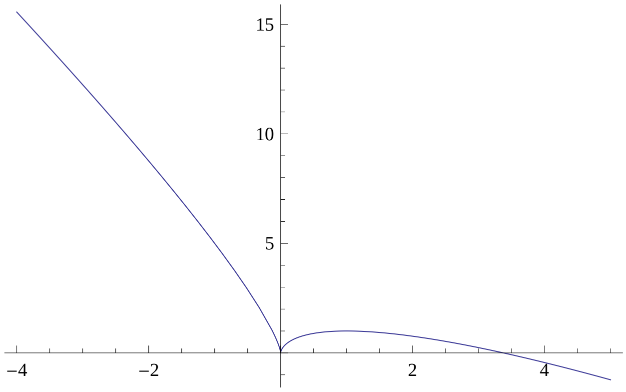
(i) For $g(x)$, first get $g'(x) = 2x^{-1/3} - 2 = \dfrac{2(1-\sqrt[3]{x})}{\sqrt[3]{x}}$. Two CP—one at $x = 0$ (since $g'(0)$ DNE there) and one where $g'(x) = 0$, namely, at $x = 1$.

(a) The factored form makes it easier to see that g' changes signs precisely at $x = 0$ and $x = 1$, and that $g'(x) < 0$ for $x < 0$ and $x > 1$, so that's where g is decreasing; g is inc on $(0, 1)$.

(b) From the above, there is clearly a local min at $x = 0$, a local max at $x = 1$.

(c) Another derivative gives $g''(x) = -(2/3)x^{-4/3}$, which is always negative (when $x \neq 0$). So g is concave down on $(-\infty, 0)$ and on $(0, \infty)$.

(d) For a graph, it won't kill you to calculate the x-intercepts, which are at $x = 0$ and $x = 27/8 = 3.375$. Then you can come up with something like the following. (This small interval does not tell the whole story; you can ask in class what I mean by that.)



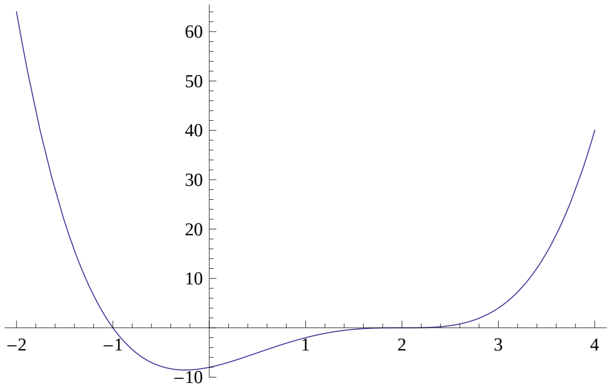
(ii) For $h(x)$ we have $h'(x) = 3(x-2)^2(x+1) + (x-2)^3(1) = (x-2)^2(3(x+1)+(x-2)) = (x-2)^2(4x+1)$. This gives two CP, at $x = 2$ and $x = -1/4$.

(a) h' changes sign only at $x = -1/4$, since the factor $(x-2)^2$ never changes sign. It follows that $h'(x) > 0$ (h is inc) for $x > 2$ and $-1/4 < x < 2$ and $h'(x) < 0$ (h is dec) for $x < -1/4$.

(b) Relative min at $x = -1/4$.

(c) $h''(x) = 2(x-2)(4x+1)+(x-2)^2(4) = (x-2)(2(4x+1)+4(x-2)) = (x-2)(12x-6)$. Thus h is concave up for $x > 2$ and $x < 1/2$, conc down for $1/2 < x < 2$.

(d) Like so:



⋆ ⋆ ⋆ Extras ⋆ ⋆ ⋆

Each st⋆rred problem is extra credit and each ⋆ is worth 5 points.

(Feel free to do these on the back of the previous page or elsewhere. Just tell me where to look.)

A. (⋆) Evaluate $\dfrac{d}{dx}\cosh(x^2 + x)$.

B. (⋆) A population of *Calcucoccus disorientus* bacteria are growing exponentially. The population triples every 7 hours. Given an intial population of one million *C. disorientus*, write a formula for the number at time t (in hours). When are there two million of these vile cooties?

C. (⋆) Find the absolute maximum and minimum values of the function $L(x) = x - \ln x$ on the interval $\left[\frac{1}{2}, \frac{3}{2}\right]$. (The trick is to do this without a calculator. You must explain your answers — not much credit otherwise. The *better* the explanation, the more credit given. A hint, if you can make sense of it: there is some symmetry in the numbers you must "test" but no symmetry in the derivative. I've said too much...)

D. (⋆ · · · ⋆) Surely I forgot something you were ready for. Ask a question you wish I had asked and answer it. Points may vary. Offer void where prohibited by law.