Calc II

MATH 222

Fall 2007

Name: ____________________

Final Exam

Instructions: Write answers to problems *on separate paper.* You may NOT use calculators or any electronic devices or notes of any kind. Loads of points are possible on the exam, but the highest grade that I will award is 115 points.

1. (5 points) Consider the area under the curve $y = 1/x^3$, where $x \geq 1$. A vertical line $x = b$ splits this region into two equal areas. Find the value of b.

2. (4 points) Find the average value of the function $y = \cos x$ over the interval $[-\pi/2, \pi/2]$. (Evaluate completely to get an exact numerical answer.)

3. (4 points each) Evaluate each of the following integrals.

   (a) $\displaystyle\int x^2 e^{3x}\, dx$

   (b) $\displaystyle\int \arctan(3x)\, dx$

4. (4 points each) Evaluate each of the following integrals.

   (a) $\int \sin^6 x\ \cos^3 x\, dx$

   (b) $\int \sec^3(2x)\ \tan^3(2x)\, dx$

5. (4 points each) Evaluate the integrals.

   (a) $\displaystyle\int \frac{dx}{\sqrt{9-4x^2}}$

   (b) $\displaystyle\int \frac{dx}{x(9+4x^2)}$

   (c) $\displaystyle\int \frac{2x^3 - x^2 - 6x + 1}{x-1}\, dx$

6. (4 points each) Evaluate each definite integral completely and simplify. If the integral diverges, say so and state how it diverges (e.g., to ∞ or $-\infty$). (Very little credit will be given for answers with no work or incorrect reasoning.)

   (a) $\displaystyle\int_{-2}^{2} \frac{dx}{x^2}$

   (b) $\displaystyle\int_{0}^{\infty} \frac{x}{x^4+1}\, dx$

7. (3 points each) The portion of the curve $y = \tan 2x$, $-\pi/6 \le x \le \pi/6$ is rotated about each of the following axes. Calculate the areas of each surface of revolution formed. (Set up the integrals.)

   (a) the y-axis

   (b) the x-axis

   (c) the line $x = \pi/2$

8. (8 points) Find the centroid $(\bar{x}, \bar{y})$ of the region $\mathcal{R}$ bounded by $y = x(x-1)$ and $y = 2$. Don't merely set this one up — calculate it completely and simplify. And you need to get this region right, as the next few problems use it, too.

9. (4 points) Let $\mathcal{R}$ be the region described in the previous problem. Find the volume of the solid of revolution formed by rotating $\mathcal{R}$ about the line $x = -1$. Set up the integral completely but don't evaluate it.

10. (4 points) Let $\mathcal{R}$ be the region used in the previous problems. Find the volume of the solid of revolution formed by rotating $\mathcal{R}$ about the line $y = -1$. Again just set up the integral.

11. (6 points) Let $\mathcal{R}$ be the region used in the previous problems. Consider the intersections of vertical lines in the xy-plane (lines parallel to the y-axis) with the region. (The region is is the union of such segments.) Treating each of these line segments as a base, erect squares perpendicular to the xy-plane on each segment. Find the volume of the solid formed by the union of those squares. Just set up the integral (although it isn't very difficult to evaluate and simplify it).

12. (4 points) State the definition of a convergent sequence. (Hint: It has an epsilon in it.)

13. (3 points) State (the hypotheses and conclusions of) the monotone convergence theorem.

14. (2 points each) Determine whether each of the following **sequences** $\{a_n\}$ converges or diverges for the given a_n. If it converges, find its limit. If it diverges to $+\infty$, say so. If it diverges to $-\infty$, say so. If it diverges in some other way, say how. No credit for "diverges" or "converges", but no extra penalties for incorrect answers.

   (a) $a_n = \dfrac{5n^3}{3(5^n)}$

   (b) $a_n = \dfrac{3(5^n)}{5(3^n)}$

   (c) $a_n = \dfrac{n^2 + \sin n}{2n + 3n^2}$

   (d) $a_n = \arctan\left(\dfrac{n^2+n}{n^2}\right)$

15. (3 points) State the precise hypotheses and conclusion of the theorem we call "the Integral Test".

16. (3 points) State the precise hypotheses and conclusion of the theorem we call "the Root Test".

17. (2 points for each correct answer, -1 for each incorrect answer, no penalty for blanks) Determine whether each of the following series is **abs**olutely convergent, **cond**itionally convergent or **div**ergent. (Also, see extra-credit problem (**??**) for a related question.)

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{2n}{3n^3+1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n(n+1)}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{e^n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \cos\frac{1}{n^2}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{\sqrt[n]{n}}$

(g) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{3^{2n}}{2^{3n}}$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(\ln n)^3}{n}$

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sin n}{(n+1)^2}$

(j) $\displaystyle\sum_{n=1}^{\infty} \arctan(1/n^2)$

(k) $\displaystyle\sum_{n=1}^{\infty} 1/\arctan(n^2)$

18. (3 points each) Find the *intervals* of convergence of each of the power series. (I.e., state all real values of x for which the series converges.)

   (a) $\displaystyle\sum_{n=1}^{\infty} n(2x)^n$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{3n^2}$

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$

   (d) $\displaystyle\sum_{n=0}^{\infty} n!(x-3)^{2n}$

19. (3 points each) Write down (you needn't derive it if you can just write it) the Maclaurin series expansion (Taylor series centered at $c = 0$) for each of the functions below. You may write four nonzero terms for each or for one extra point each, write the series in closed (summation) form.

   (a) e^{-x}

   (b) $\displaystyle\frac{1}{3+2x}$

   (c) $\cos 3x$

   (d) $x \sin(x^2)$

   (e) $\displaystyle\frac{1}{(1-x)^2}$

20. (3 points) Give the first four nonzero terms of the Taylor series for the following. Again, you needn't derive it if you can just write it.

   (a) x^2, centered at $c = 1$

   (b) e^x, centered at $c = 1$

$\star\star\star\star$ Extras $\star\star\star\star$

(5 points per $\star$)

A.) ($\star$) Use a theorem of Pappus to evaluate the volume of the solid obtained by rotating the figure ABC about the line $x = 1$, where ABC is the triangle with vertices $A = (2, 3),\ B = (2, 5),\ C = (5, 5)$.

B.) ($\star$) Integrate: $\int \sin x\ \cos 3x\ dx$.

C.) ($\star$) Write four terms of the binomial expansion for $\sqrt[4]{1 + x}$.

D.) ($\star$) Write the repeating decimal $.0246666666666\cdots$ as a ratio of integers.

E.) ($\star$) Write the first three nonzero terms for the Maclaurin series of $\cos x$ and $\sin x$. Give the first three nonzero terms of the power series for $\sin x \cos x$ by directly sufficiently many terms of the two series. Then get he result a cooler way, if you can. Also, use long division of series to get the first three nonzero terms $\sec x$ by dividing the series for $\cos x$ into the constant 1. I suppose you could check your answer by using Taylor's formula for the Maclaurin series of $\sec x$ (but I wouldn't want to).

F.) ($\star$) Maybe you can evaluate the following limit.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sin\left(\frac{n}{N}\right)$$

G.) ($\star$) Don't try this at home. No — DO try it at home, but don't try it here. Show that the area under the curve $y = \tan x$ from 0 to $\pi/2$ is infinite. Then show that the area under the curve $y = \sqrt{\tan x}$ from 0 to $\pi/2$ is equal to $\pi/\sqrt{2}$. It aint that easy. Send me email for hints. I'll post this final exam online. (Remind me if I forget).

H.) ($\star\cdots\star$) Ask a question you wish I had asked and answer it. Points may vary. Offer void where prohibited by time!