Calc II | Spring 2009

MATH 222 | Name:

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Final Exam

Instructions: Write answers to problems *on separate paper.* You may NOT use calculators or any electronic devices or notes of any kind. Loads of points are possible on the test and there is a ton of extra credit, worth 4 points per $\star$. No maximum. Go for it!

When specified, you may leave **definite** integrals unevaluated ("just set it up"), but they must be "ready to evaluate," that is, each must be a definite integral involving one variable only (e.g., no mixed x's and y's) with **explicit functions and explicit limits of integration and no uses of |absolute values| whatsoever**.

1. (6 points) Find the total area bounded by $y = x^3$ and $y = x^2 + 2x$. Evaluate completely
2. Consider the region $\mathcal{R}$ bounded by the lines $y = 2x$, $y = x + 2$, and $x = -1$.
   (a) (5 points) Use the method of shells to find the volume of the solid obtained by rotating $\mathcal{R}$ about the line $x = -2$. Evaluate and simplify completely.
   (b) (3 points) Same as part (a) but use the method of disks/washers. (Set up the integral(s).)
   ($\star$) Use a Theorem of Pappus to check the result of part (a).
   ($\star$) Evaluate the integral(s) in part (b) to check the answer in (a). Show all *your* steps, but feel free to work smartly—there are ways to reduce the number of menial computations if you think about it. Or just slog through if you're more comfortable with that.
3. (4 points) Find the average value of the function $y = \sqrt{9 - x^2}$ on the interval $[0, 3]$. (Write the integral but also give the exact, simplified, numerical answer.)
4. (2 points) State the precise hypotheses and conclusion of the Mean Value Theorem for Integrals.
5. (6/5/4/3/3 points) Evaluate each of the following integrals.
   (a) $\int xe^{5x}\,dx$
   (b) $\int x\ln(7x)\,dx$
   (c) $\int \tan^4(x)\,dx$
   (d) $\displaystyle\int \frac{x^3}{\sqrt{9 + x^2}}\,dx.$
   (e) $\displaystyle\int \frac{x^3}{x^2 - x - 2}\,dx$

6. (5/4/2 points) Consider the portion of the parabola $y = 4x(1-x)$ that lies above the x-axis. Find the surface area obtained when this curve is

   (a) rotated about the y-axis,

   (b) rotated about the x-axis,

   (c) rotated about the line $x = -1$.

   (Set up the integrals.)

7. (5 points) Use integrals to find the centroid $(\bar{x}, \bar{y})$ of the region described in problem #2. (For half credit, set up the integrals. For full credit, evaluate the integrals, showing all work, and find the actual point. You could check your answer with an easy formula for the centroid of a triangle.)

8. (1 point each) Determine whether each of the following **sequences** $\{a_n\}$ converges or diverges. If it converges, find its limit. If it diverges to $+\infty$, say so. If it diverges to $-\infty$, say so. If it diverges in some other way, say how. No credit for "diverges" or "converges", but no penalties for incorrect answers.

   (a) $\left\{\dfrac{2n}{\ln(n+1)}\right\}$

   (b) $\left\{\dfrac{2n^3+n}{e^{n+1}}\right\}$

   (c) $\left\{\dfrac{n!(2n^2-1)}{(n+2)!}\right\}$

   (d) $\left\{\dfrac{n^n}{2^n}\right\}$

   (e) $\left\{(-1)^n\dfrac{2^{3n}}{3^{2n}}\right\}$

   (f) $\left\{\arcsin\left(\dfrac{n}{2n+1}\right)\right\}$

   (g) $\left\{\arctan\left(\dfrac{n^2}{1-n}\right)\right\}$

   (h) $\left\{(-1)^n\dfrac{2^n}{n^2}\right\}$

   (i) $\left\{\dfrac{n}{(3n+1)\sqrt{4n^2+n}}\right\}$

   (j) $\left\{\sqrt{n^2+3n}-n\right\}$

9. (2 points) State the precise hypotheses and conclusion of the theorem we call "the Alternating Series Test".

10. (2 points) State the precise hypotheses and conclusion of the theorem we call "the Ratio Test".

11. (6 points) Explain fully the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty}\left(\arccos\left(\frac{n}{2n+1}\right)\right)^n$.

12. (1 point for each correct answer, $-1/2$ for each incorrect answer, no penalty for blanks) Determine whether each of the following series is **abs**olutely convergent, **cond**itionally convergent or **div**ergent. (Also, see extra-credit problem (F) for a related question.)

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{2n}{3n-1}$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{n!}$

(c) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{2^n}$

(d) $\displaystyle\sum_{n=1}^{\infty}\cos\frac{(-1)^n(n+1)}{2n^3+3}$

(e) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^e}{e^n}$

(f) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(\ln n)^3}{n^2}$

(g) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{2^{3n}}{3^{2n}}$

(h) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(\ln n)^3}{n}$

(i) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n!}{(n+1)!}$

(j) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{2^n+e}{e^n+2}$

(k) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n^n}{n!}$

13. (6/4/3/3 points each) Find the intervals of convergence of each of the power series.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(3x)^n}{n}$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(x-1)^n}{2n^2}$

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{(2x)^n}{(n+2)!}$

(d) $\displaystyle\sum_{n=0}^{\infty}(n)!(2x-3)^n$

14. (5/5/4/3/3 points each) Write down (you needn't derive it if you can just write it) the Maclaurin series expansion (Taylor series centered at $c=0$) for each of the functions below. (Write your answer in closed form (e.g., in the form $\sum_{n=0}^{\infty}a_nx^n$), or, if you are unable to do that, for one less point per sum, you may write the answer in the form $a_0+a_1x+a_2x^2+\cdots$, provided you give at least the first four *nonzero* terms.)

(a) e^{3x}

(b) $\dfrac{1}{2-3x}$

(c) $\cos x^2$

(d) $x^2\sin 7x$

(e) $\arctan 3x$

15. (5/4/4 points) Give the first four nonzero terms of the Taylor series for the following. Again, you needn't derive it if you can just write it.

(a) x^2, centered at $c=1$

(b) e^x, centered at $c=1$

(c) $\sqrt[4]{1+x}$, centered at $c=0$

$\star\star\star\star$ Extras $\star\star\star\star$

(5 points per $\star$)

A.) ($\star$) Use a theorem of Pappus to evaluate the volume of the torus obtained by rotating the figure $ABCD$ about the line $y = 4$, where $ABCD$ is the rectangle with vertices

$$A = (2, 3),\ B = (5, 2),\ C = (3, -4),\ D = (0, -3).$$

B.) ($\star$) Consider the region described in problem #2. Take it to be the base of a solid whose cross-sections parallel to the y-axis are semicircles. Find the volume of the solid. (Set it up.)

C.) ($\star$) What is the length of the parameterized curve given below? (Set up the integral.)

$$x = 2t + 1, \quad y = t + \sin 3\pi t, \quad 0 \le t \le 4$$

(The graph of this one is kinda cool.)

D.) ($\star\cdots\star$) There are several of the convergent series in problems #12 and #13 whose exact sum can be found using tricks we covered. I'll give 5 points for each one you can evaluate exactly. (Be mindful of the lower indices in the sums; sometimes you'll need to make a minor adjustment to a typical formula because of that. As an example, and for +3 easy points, first show that $\displaystyle\sum_{n=2}^{\infty} \frac{2^n}{n!} = e^2 - 3$.)

E.) ($\star$) Evaluate the sum $\dfrac{1}{2} + \dfrac{1}{30} + \dfrac{1}{400} + \dfrac{1}{5000} + \cdots$.

F.) ($\star$) Give the terms up to degree four of the power series obtained by expanding the following

$$(x + 2x^2 + 3x^3 + 4x^4 + \cdots)(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots)$$

and give the terms up to degree two obtained by dividing (using "infinite long division")

$$\frac{x + 2x^2 + 3x^3 + 4x^4 + \cdots}{x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots}.$$

G.) ($\star$) What point lies $8/17$ the way from the point $(1, 2)$ to the point $(-3, 5)$?

H.) ($\star$) Write the repeating decimal $.0123232323232323\cdots$ as a ratio of integers.

I.) ($\star$) Write parameterizations for each of the curves described below. Be sure to give precise intervals for the parameter (e.g., $t \in [0, 2]$ or $0 \le \theta < 2\pi$, etc.).

(a) The line segment joining the points $(0, 3)$ and $(3, 0)$.

(b) The bottom right quarter of the circle of radius 3 and centered at $(-3, 0)$

J.) ($\star\cdots\star$) Ask a question you wish I had asked and answer it. Points may vary. Offer void where prohibited by time!