Calc II
MATH 222

Fall 2005
Name: ____________

T1

Instructions: Write answers to problems *on separate paper.* You may NOT use calculators or any electronic devices or notes of any kind. Each st⋆rred problem is extra credit and each ⋆ is worth 5 points. (These are just more problems, but harder. They're worth fewer points so that you're not unduly tempted.) Loads of points are possible on the test, but the highest grade that I will award is 115 points.

Unless otherwise specified, you may leave **definite** integrals unevaluated ("just set it up"), but they must be "ready to evaluate," that is, each must be a definite integral involving one variable only (e.g., no mixed x's and y's) with **explicit functions and explicit limits of integration and no absolute values whatsoever**. However, if you would like to evaluate the integrals, they are worth an extra 4 points each. (*Warning:* Save these for last; some may be too difficult.)

Diminishing returns: Phrases such as "8/6/4 points" (see problem #3, for instance) refer to the points awarded for doing several parts of a problem. The example here indicates that 8 points will be awarded if any one problem of the three is done correctly, $8 + 6$ points if any two are correct, $8 + 6 + 4$ points for all three.

1. Find the exact areas of the regions described below. (Do not merely write integrals in this one; evaluate completely to find the exact numerical result.)

   (a) (5 points) the region bounded by the curves $y = x^2 - x$ and $y = x + 1$

   The curves intersect when $x^2 - x = x + 1$, so $x^2 - 2x - 1 = 0$. This doesn't factor (sorry about that), so use the quadratic formula to get $x = 1 \pm \sqrt{2}$. To see which curve is on top, a quick sketch would suffice, or you can test an easy value inside the interval. The most obvious choice (to me) is $x = 1$ but $x = 0$ is also good, which shows that the line lies above the parabola in our region. You can also simply note that the quadratic opens upward, so it has to be highest to the *right* of the region. The area is therefore

$$
\begin{aligned}
A &= \int_{\text{Left}}^{\text{Right}} (\text{Top} - \text{Bottom})dx \\
&= \int_{1-\sqrt{2}}^{1+\sqrt{2}} ((x+1) - (x^2 - x))dx = \int_{1-\sqrt{2}}^{1+\sqrt{2}} (-x^2 + 2x + 1)dx \\
&= \left(-\frac{x^3}{3} + x^2 + x\right)\Bigg|_{1-\sqrt{2}}^{1+\sqrt{2}} \\
&= \left(-\frac{(1+\sqrt{2})^3}{3} + (1+\sqrt{2})^2 + (1+\sqrt{2})\right) - \left(-\frac{(1-\sqrt{2})^3}{3} + (1-\sqrt{2})^2 + (1-\sqrt{2})\right).
\end{aligned}
$$

This is an exact numerical form, so *stop*. If you were to simplify it, you'd get $A = 4\sqrt{2}/3$. It isn't really difficult to simplify this using the obvious, but tedious methods. (We did one in class that was much worse.) There are also little tricks (that you know) which make some of it easier than the drudgery of expanding and combining like terms. And there is even a very cute algebraic trick that makes the answer pop out with a very easy calculation (the trick uses polynomial long division). If you ever want to see it, let me know.

(b) (10 points) the region bounded by the curves $y = x^3 - x^2$ and $y = x^2 + x$

You'll quickly notice (perhaps with horror) that these functions are the same as in part (a) but multiplied by x. The good news is that when you set them equal and solve, you end up solving almost the exact equation as before. That's also the bad news. But not TOO bad:

$$x^3 - x^2 = x^2 + x \;\Rightarrow\; x^3 - 2x^2 - x = 0 \;\Rightarrow\; x(x^2 - 2x - 1) = 0,$$

so the intersections are at $x = 0, 1 \pm \sqrt{2}$. It is clear that

$$1 - \sqrt{2} < 0 < 1 + \sqrt{2},$$

so you can see how to split the interval. Now which curve is on top in each subinterval? The graph is harder to sketch, but again you can check $x = 1$ or you can note that the cubic must eventually (as $x \to \infty$) be above the quadratic. Putting this together gives

$$\begin{aligned} A &= \int_{1-\sqrt{2}}^{0} (\text{cubic} - \text{quadratic})dx + \int_{0}^{1+\sqrt{2}} (\text{quadratic} - \text{cubic})dx \\ &= \int_{1-\sqrt{2}}^{0} (x^3 - 2x^2 - x)dx + \int_{0}^{1+\sqrt{2}} (-x^3 + 2x^2 + x)dx \\ &= \left(\frac{x^4}{4} - \frac{2x^3}{3} - \frac{x^2}{2}\right)\Bigg|_{1-\sqrt{2}}^{0} + \left(-\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2}\right)\Bigg|_{0}^{1+\sqrt{2}}, \end{aligned}$$

and you're done. (The simplified version is surprisingly nice: $A = 23/6$. Again, there is a relatively calculation-free way to get this rather quickly, but it requires an algebraic trick.)

2. (10 points) Find the area bounded by $y = |x - 1|$ and $y = x^2 - 2$. (Set it up. Warning: you'll need the quadratic formula to find the limits. But you won't be evaluating the integral(s) so that part is no big deal.)

Even a crude sketch of the graphs show that you get a single region. The absolute value function gives a "vee" opening upward at 45° angles from the point $(1, 0)$; the quadratic is a parabola opening upward from $(0, -2)$. The left ray of the absolute value graph is given by $y = -x + 1$, the right ray is $y = x - 1$. So the left ray intersects the parabola where $-x + 1 = x^2 - 2$. The quadratic formula gives $x = \dfrac{-1 \pm \sqrt{13}}{2}$ and it is clear that you need the leftmost root, $x = \dfrac{-1 - \sqrt{13}}{2}$. Similarly, the right ray hits the parabola where $x - 1 = x^2 - 2$ and you get $x = \dfrac{1 + \sqrt{5}}{2}$. The absolute value function is on top in this interval, so

$$A = \int_{(-1-\sqrt{13})/2}^{1} ((-x + 1) - (x^2 - 2))dx + \int_{1}^{(1+\sqrt{5})/2} ((x - 1) - (x^2 - 2))dx.$$

3. (10/6/3 points) Use the method of disks/washers to find the volumes of the solids obtained by rotating the region bounded by the curves $y = x^2$ and $y = 2x$ about each of the following lines. (Set up the integrals.)

This region is easy — the curves intersect at $x = 0, 2$ with the line on top.

(a) the x-axis

Rotating about the x-axis with disks or washers means the we need to integrate on x. For each x in $[0, 2]$ the slice gives a washer with inner radius $y = x^2$ and outer radius $y = 2x$. So

$$\begin{aligned} V &= \int_{\text{left}}^{\text{right}} \pi(r_{\text{outer}}^2 - r_{\text{inner}}^2)dh \\ &= \int_0^2 \pi((2x)^2 - (x^2)^2)dx. \end{aligned}$$

(If you actually integrate this you get $64\pi/15$.)

(b) the y-axis

This time we need to integrate on y. The bounds give us the interval from $y = 0$ to $y = 4$. For each such y the slices give an inner radius of $x = y/2$ and an outer radius of $x = \sqrt{y}$. We get

$$V = \int_0^4 \pi((\sqrt{y})^2 - (y/2)^2)dy.$$

The actual volume turns out to be $8\pi/3$.

(c) the line $y = -1$

This is just like part (a), but the axis is one unit lower. So the radii are one unit greater:

$$V = \int_0^2 \pi((2x + 1)^2 - (x^2 + 1)^2)dx.$$

4. (10/4 points) Use the method of shells to find the volume of the solid obtained by rotating the region bounded by $y = x^2 - 2x + 3$ and $y = x + 3$ about each of the following lines. (Set up the integrals.) This gives the region between $x = 0$ and $x = 3$, with the line on top.

(a) the y-axis

For shells about the y-axis we integrate on x. For each $x \in [0, 3]$ we get a shell whose radius is x and whose height is $(x + 3) - (x^2 - 2x + 3) = 3x - x^2$. We get

$$V = \int_{\text{left}}^{\text{right}} 2\pi rhdr = \int_0^3 2\pi x(3x - x^2)dx.$$

The actual volume turns out to be $27\pi/2$.

(b) the line $x = -2$

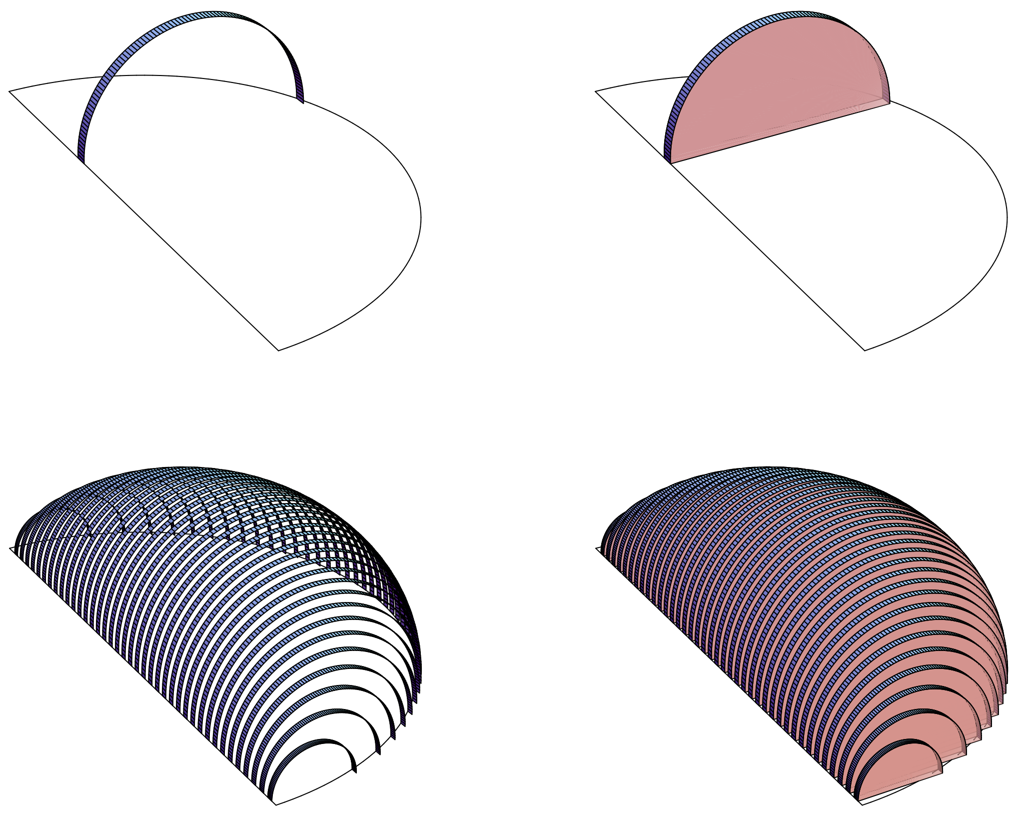
Same as above but now the axis of rotation is 2 units farther away, so the radii are increased by 2:

$$V = \int_0^3 2\pi(x+2)(3x - x^2)dx.$$

The result is $63\pi/2$.

5. (10 points) A solid's base is a semicircle $\mathcal{S}$ of radius R. (If R scares you, let $R = 2$, but it will cost you three points.) Cross-sections perpendicular to this base *and perpendicular to the base of* $\mathcal{S}$ are also semicircles with their own bases in $\mathcal{S}$. Find the volume of the solid. (Evaluate the integral in this one.)

Here's a picture.



For simplicity I'll let the semicircle $\mathcal{S}$ of radius R sit in the xy-plane and have equation $y = \sqrt{R^2 - x^2}$. For each $x \in [-R, R]$, a semicircular cross-section of the solid has as its base the segment joining the points $(x, 0)$ and $(x, \sqrt{R^2 - x^2})$ in the xy-plane. This is a diameter of the cross-section, so its radius, call it $r(x)$, is half of that, i.e., $r(x) = \frac{1}{2}\sqrt{R^2 - x^2}$. So the area of the cross-section is $\frac{1}{2}\pi(r(x))^2 = \frac{\pi}{8}(R^2 - x^2)$. Now we have it:

$$
\begin{aligned}
V &= \int_{-R}^{R} \frac{\pi}{8}(R^2 - x^2)dx = \int_{0}^{R} \frac{\pi}{4}(R^2 - x^2)dx \\
&= \frac{\pi}{4}(R^2x - x^3/3)\Big|_0^R \\
&= \frac{\pi}{4} \cdot \frac{2R^3}{3} \\
&= \frac{\pi R^3}{6}.
\end{aligned}
$$

6. (7 points) Find the average value of the function $y = \sin x$ on the interval $[0, \pi]$. (Evaluate completely to get an exact numerical answer.)

   A quickie, by the formula:

$$
\begin{aligned}
\bar{f} &= \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\pi}\int_0^\pi \sin x\,dx \\
&= \frac{1}{\pi}(-\cos x)\Big|_0^\pi = \frac{1}{\pi}(1 - (-1)) \\
&= \frac{2}{\pi}
\end{aligned}
$$

7. (7 points) State precisely the Mean Value Theorem for Integrals. (Completeness counts.)

   (Look it up.)

8. (8/6/4 points) Evaluate each of the following integrals. (I'd use integration by parts.)

(a) $\int x^2 e^{3x}\, dx$

Two integrations by parts gives

$$\int x^2 e^{3x}\, dx = \cdots = \frac{1}{27} e^{3x} \left(9x^2 - 6x + 2\right) + C.$$

(b) $\int \arctan(3x)\, dx$

An integration by parts (with $u = \arctan x$ and $dv = dx$) gives

$$\int \arctan(3x)\, dx = \cdots = x \arctan(3x) - \frac{1}{6} \ln\left(9x^2 + 1\right) + C.$$

(c) $\int \sqrt[3]{x}\, \ln x\, dx$

Let $u = \ln x$ and $dv = \sqrt[3]{x}\, dx$. Then $du = dx/x$ and $v = 3x^{4/3}/4$ and we get

$$\begin{aligned} \int \sqrt[3]{x}\, \ln x\, dx &= \ln x \, \frac{3x^{4/3}}{4} - \int \frac{3x^{4/3}}{4} \frac{1}{x}\, dx \\ &= \ln x \, \frac{3x^{4/3}}{4} - \int \frac{3x^{1/3}}{4}\, dx \\ &= \ln x \, \frac{3x^{4/3}}{4} - \frac{9x^{4/3}}{16} + C \end{aligned}$$

9. (8/6/4 points) Evaluate the following ~~intervals~~ integrals. (These might be quickies if you remember certain formulas. Otherwise you need a trick or two.)

(a) $\int \ln(5x)\, dx$

Since I remember that $\int \ln u\, du = u \ln u - u + C$, I use my linear-substitution-trick to get

$$\int \ln(5x)\, dx = \frac{1}{5}\,(\,(5x)\ln(5x) - 5x\,) + C.$$

(b) $\int \sec(5x)\, dx$

Since $\int \sec u = \ln|\sec u + \tan u| + C$, we get

$$\int \sec(5x)\, dx = \frac{1}{5}\,(\,\ln|\sec 5x + \tan 5x|\,) + C.$$

(c) $\int \sec^3(5x)\, dx$

You either have to know the "folded IbP" trick (and be able to do it) or just know the formula:

$$\int \sec^3 u\, du = \frac{1}{2}(\,\sec u\ \tan u + \ln|\sec u + \tan u|\,) + C,$$

and in either case you get

$$\int \sec^3 5x\, dx = \frac{1}{10}\,(\,\sec 5x\ \tan 5x + \ln|\sec 5x + \tan 5x|\,) + C,$$

10. (8/6/4 points) Evaluate each of the following ~~intervals~~ **integrals!** (I'd use some trig identities.)

(a) $\int (1 + \cos 5x)^2 \, dx$

Expand the square and use the fact that $\cos^2(5x) = \dfrac{1 + \cos 10x}{2}$. The result is that

$$\int (1 + \cos 5x)^2 \, dx = \int (\frac{3}{2} + 2\cos 5x + \frac{1}{2} \cos^2(5x))dx = \frac{3x}{2} + \frac{2}{5} \sin 5x + \frac{1}{20} \sin 10x + C.$$

(b) $\int \tan^3 x \, dx$

We did this in class:

$$\begin{aligned} \tan^3 x \, dx &= \int \tan x \cdot \tan^2 x \, dx = \int \tan x \, (\sec^2 x - 1) \, dx \\ &= \int \tan x (d \tan x) - \int \tan x \, dx \\ &= \frac{1}{2} \tan^2 x + \ln|\cos x| + C \end{aligned}$$

(c) $\int \sin^3 x \, \cos^3 x \, dx$

Noting that $\cos x \, dx = d \sin x$, we separate that and write the rest in terms of $\sin x$:

$$\begin{aligned} \int \sin^3 x \, \cos^3 x \, dx &= \int \sin^3 x \, \cos^2 x \, \cos x \, dx \\ &= \int \sin^3 x (1 - \sin^2 x) \cos x \, dx = \int (\sin^3 x - \sin^5 x) d \sin x \\ &= \frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x + C \end{aligned}$$

$\star \quad \star \quad \star$ Extras $\star \quad \star \quad \star$

Feel free to do these on the back of the previous page or elsewhere. Just tell me where to look.

A. ($\star$) Find the exact area bounded by the x-axis, the y-axis, the line $x = 2$ and the curve $y = 1 + \sqrt{4 - x^2}$.

B. ($\star\star$) Same as in problem A, but with $x = 1$ instead of $x = 2$.

C. ($\star$) Consider the region whose area you found in problem #1. Now consider the solid formed by the union of all line segments drawn from points in this region to the point 5 units above the above the xy-plane, directly above the origin. Find the volume of this solid.

D. ($\star$) Use the method of shells to find the volume of the solid obtained by rotating the region given in problem #4 about the x-axis.

E. ($\star$) Prove the Mean Value Theorem for Integerals by applying the usual Mean Value Theorem (for derivatives) to the function $\int_a^x f(t)\,dt$.

F. ($\star$) Evaluate the integral $\int \cos mx \;\cos nx\,dx$, where m and n are constants.

G. ($\star$) Find a reduction formula for the integral $\int \sec^{2n-1} x\,dx$, where n is a positive integer. The trick we used for $\int \sec^3 x\,dx$ works.

H. ($\star\star$) Here's a nice identity: $\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$.

(a) Prove that the identity is true.

(b) Use the techniques of integration we've been studying to evaluate $\int \sin^3 x\,dx$.

(c) Use the identity to find this same integral.

(d) use the equality of the integrals you found to write a similar identity for $\cos^3 x$.

I. ($\star\cdots\star$) Ask a question you wish I had asked and answer it. Points may vary. Offer void where prohibited by law.