Calc II | Fall 2005
MATH 222 | Name:

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T2

Instructions: Write answers to problems *on separate paper.* You may NOT use calculators or any electronic devices or notes of any kind. Each st⋆rred problem is extra credit and each ⋆ is worth 5 points. (These are just more problems, but harder. They're worth fewer points so that you're not unduly tempted.) Loads of points are possible on the test, but the highest grade that I will award is 115 points.

In specified cases you may leave **definite** integrals unevaluated (I'll say, "just set it up"), but these must then be "ready to evaluate," that is, each must be a definite integral involving one variable only (e.g., no mixed x's and y's) with **explicit functions and explicit limits of integration and no absolute values whatsoever**. However, if you would like to evaluate the integrals, they are worth an extra 4 points each. (*Warning:* Save these for last; some may be too difficult or impossible.)

Diminishing returns: Phrases such as "8/6/4 points" refer to the points awarded for doing several parts of a problem. The example here indicates that 8 points will be awarded if any one problem of the three is done correctly, $8 + 6$ points if any two are correct, $8 + 6 + 4$ points for all three.

1. (10 points) Evaluate the integral $\int \sqrt{9-4x^2}\,dx$.

   A trig substitution of $4x^2 = 9\sin^2\theta$, i.e., $x = \frac{3}{2}\sin\theta$, gives $dx = \frac{3}{2}\cos\theta\,d\theta$ and $\sqrt{9-4x^2} = 3\cos\theta$. This converts the integral to

$$\begin{aligned}
\int \frac{9}{2}\cos^2\theta\,d\theta &= \frac{9}{2}\int \frac{1+\cos 2\theta}{2}\,d\theta \\
&= \frac{9}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C \\
&= \frac{9}{4}\left(\theta + \sin\theta\ \cos\theta\right) + C \\
&= \frac{9}{4}\left(\arcsin\frac{2x}{3} + \frac{2x}{3}\sqrt{1-\left(\frac{2x}{3}\right)^2}\right) + C \quad \checkmark \\
&= \frac{9}{4}\left(\arcsin\frac{2x}{3} + \frac{2x}{9}\sqrt{9-4x^2}\right) + C \\
&= \frac{9}{4}\arcsin\frac{2x}{3} + \frac{x}{2}\sqrt{9-4x^2} + C.
\end{aligned}$$

2. (10/8/6 points) Evaluate the integrals.

(a) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-4}}$

See #16 in 7-3 for a similar problem. Let $x = 2\sec\theta$ to convert this to

$$\begin{aligned}\int \frac{2\sec\theta\ \tan\theta\ d\theta}{4\sec^2\theta\ 2\tan\theta} &= \frac{1}{4}\int \cos\theta\ d\theta \\ &= \frac{1}{4}\sin\theta + C \\ &= \frac{1}{4}\cdot\frac{\sqrt{x^2-4}}{x} + C\end{aligned}$$

(b) $\displaystyle\int \frac{dx}{(x^2+2x+2)^2}$

(This is #27 in 7-3.) Complete the square to write $x^2+2x+2 = (x+1)^2+1$ and then let $x+1 = \tan\theta$. You end up with $\displaystyle\frac{1}{2}\left(\frac{x+1}{x^2+2x+2} + \arctan(x+1)\right) + C$.

(c) $\displaystyle\int \frac{dx}{\sqrt{5-3x^2}}$

This is a quickie — just notice that $\sqrt{5-3x^2} = \sqrt{3}\sqrt{\frac{5}{3}-x^2}$ and use the formula

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin(u/a) + C,$$

with $a = \sqrt{5/3}$. Or do a trig substitution, if you must. Either way, you should get $\displaystyle\frac{\arcsin\left(\sqrt{\frac{3}{5}}x\right)}{\sqrt{3}} + C$.

3. (10 points) Evaluate the integral $\displaystyle\int \frac{dx}{x^3-4x}$.

Bust it up with partial fractions.

$$\frac{1}{x^3-4x} = \frac{1}{x(x-2)(x+2)} = \cdots = \frac{1}{8(x+2)} + \frac{1}{8(x-2)} - \frac{1}{4x}.$$

Integrate to get

$$\frac{\ln|x+2|}{8} + \frac{\ln|x-2|}{8} - \frac{\ln|x|}{4} + C.$$

4. (12/8/5 points) Evaluate the integrals.

(a) $\displaystyle\int \frac{x^3 - 2x + 1}{(x+1)(x+2)}\, dx$

Use long division to divide, getting

$$\frac{x^3 - 2x + 1}{(x+1)(x+2)} = x - 3 + \frac{5x+7}{(x+1)(x+2)}.$$

Then use partial fractions to write this as

$$x - 3 + \frac{2}{x+1} + \frac{3}{x+2},$$

which is easily integrated to get

$$\frac{1}{2}x^2 - 3x + 2\ln|x+1| + 3\ln|x+2| + C.$$

(b) $\displaystyle\int \frac{x^2}{(x+1)^2(x+2)}\, dx$

Use a partial fraction decomposition of the form $\dfrac{A}{x+2} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$ to rewrite as

$$\int \left(\frac{4}{x+2} - \frac{3}{x+1} + \frac{1}{(x+1)^2} \right) dx = 4\ln|x+2| - 3\ln|x+1| - \frac{1}{x+1} + K.$$

(c) $\displaystyle\int \frac{x^3}{(x^2+1)^2}\, dx$

This becomes

$$\int \left(\frac{x}{x^2+1} - \frac{x}{(x^2+1)^2} \right) dx = \frac{1}{2}\left(\ln\left(x^2+1\right) + \frac{1}{x^2+1} \right) + C.$$

5. (5/5/3/3/3 points) Evaluate each definite integral completely and simplify. If the integral diverges, say so and state how it diverges (e.g., to ∞ or $-\infty$).

(a) $\displaystyle\int_0^1 \frac{dx}{x^2}$

It diverges to ∞. (We saw that $\displaystyle\int_0^a \frac{dx}{x^p} = \infty$ for $p \geq 1$ and $a > 0$.)

(b) $\displaystyle\int_8^\infty \frac{dx}{x^{8/3}}$

This converges to $3/160$.

(c) $\displaystyle\int_{-1}^8 \frac{dx}{x^{8/3}}$

This diverges to $+\infty$ for the same reason as in part (a). It must first be noticed that the integrand is discontinuous at $x = 0$, which is in the interval of integration, hence the integral should be viewed as a sum of two improper integrals: $\displaystyle\int_{-1}^0 \frac{dx}{x^{8/3}} + \int_0^8 \frac{dx}{x^{8/3}}$, both parts of which go to ∞.

(d) $\displaystyle\int_{-3}^{0} \frac{dx}{x^2-4}$

The integrand is discontinuous at $x = -2$ in the interval of integration, so again it is a sum of two improper integrals: $\displaystyle\int_{-3}^{-2} \frac{dx}{x^2-4} + \int_{-2}^{0} \frac{dx}{x^2-4}$. Since $\displaystyle\frac{1}{x^2-4} = \frac{1}{4(x-2)} - \frac{1}{4(x+2)}$, the second term of which diverges on each subinterval. While it is true that on the left subinterval the divergence is to $-\infty$ and on the right it is to $+\infty$, according to our definition of such improper intergrals, the result is that the integral diverges.

(e) $\displaystyle\int_{-\infty}^{0} \frac{e^x\,dx}{e^{2x}+1}$

Using the substitution $u = e^x$, and noting that $u = 1$ when $x = 0$ and $u \to 0$ as $x \to -\infty$, this becomes

$$\int_0^1 \frac{du}{u^2+1} = \arctan 1 - \arctan 0 = \pi/4.$$

6. (10 points) Calculate the arc length of the curve $y = \ln x$ from $x = 1$ to $x = e$. (Set it up.)

$L = \int_1^e \sqrt{1 + (1/x)^2}\,dx$. (DONE!)

If you want to evaluate this one for extra credit, you'd get

$$\int \sqrt{1+(1/x)^2}\,dx = \ln\left(\frac{x}{\sqrt{x^2+1}+1}\right) + \sqrt{x^2+1} + C.$$

Taking it between the given limits yields

$$1 - \sqrt{2} + \sqrt{1+e^2} + \ln\left(1+\sqrt{2}\right) - \ln\left(1+\sqrt{1+e^2}\right).$$

7. (12/8/4/4/4/4 points) Consider the portion of the curve $y = \sin x$, $0 \le x \le \pi$. Calculate the surface areas of each ~~solid~~ surface of revolution, where the arc is rotated about each of the following axes. (Set them up.)

(a) the y-axis

$A = \displaystyle\int_0^\pi 2\pi x\,\sqrt{1+\cos^2 x}\,dx$. We cannot handle this integral at all!

(b) the x-axis (be careful)

$A = \displaystyle\int_0^\pi 2\pi \sin x\,\sqrt{1+\cos^2 x}\,dx$. For extra credit it would take a bit of cranking, but you'd get $A = 2\pi\left(\sqrt{2} + \sinh^{-1}(1)\right)$.

(c) the line $y = 2$

Since the axis is above the curve, our "radius" $r(x)$ is, for each x, given by $r(x) = 2 - \sin x$. So the area is $\int_0^\pi 2\pi(2-\sin x)\sqrt{1+\cos^2 x}\,dx$. Fuhgeddaboutit.

(d) the line $x = -2$ (keep being careful)

Here $r(x) = x + 2$, so $A = \int_0^\pi 2\pi(x+2)\sqrt{1+\cos^2 x}\,dx$ Again, no way.

(e) the line $x = \pi/2$ (amusing but a bit tricky)

The axis of revolution lies inside the interval of revolution, so it cuts the graph! Fortunately, the curve is symmetric about the point, so we just take half:

$$A = \int_0^{\pi/2} 2\pi(\pi/2 - x)\sqrt{1+\cos^2 x}\, dx.$$

It would be incorrect to write $A = \int_0^{\pi} 2\pi(\pi/2 - x)\sqrt{1+\cos^2 x}\, dx$, which is zero!

(f) the line $y = 1/2$ (seriously)

The axis cuts the graph here, too, necessitating the chopping up of the integral, since for each x, the radius of rev'n is $r(x) = |1/2 - \sin x|$. Now as per our instructions, we must get rid of all absolute values (!), so observing that $\sin x = 1/2$ when $x = \pi/6$ and $x = 5\pi/6$, we get

$$\begin{aligned} A &= \int_0^{\pi} 2\pi|1/2 - \sin x|\sqrt{1+\cos^2 x}\, dx \\ &= \int_0^{\pi/6} 2\pi(1/2 - \sin x)\sqrt{1+\cos^2 x}\, dx \\ &\quad + \int_{\pi/6}^{5\pi/6} 2\pi(\sin x - 1/2)\sqrt{1+\cos^2 x}\, dx + \int_{5\pi/6}^{\pi} 2\pi(1/2 - \sin x)\sqrt{1+\cos^2 x}\, dx \end{aligned}$$

8. (6 points) Use the Comparison Theorem to decide if the integral

$$\int_1^\infty \frac{1+\cos(\sqrt{x})}{e^x}\,dx$$

is convergent or divergent. *Your answer will be worth very little without a clear explanation.*

We have no hope of antidifferentiating this sucker, but we can observe that for each $x \in [1, \infty)$, we have $-1 \le \cos\sqrt{x} \le 1$, hence

$$0 \le \frac{1+\cos\sqrt{x}}{e^x} \le \frac{2}{e^x}.$$

Since it is easy to show that $\int_1^\infty \frac{2}{e^x}\,dx$ converges, so too does our integral.

⋆ ⋆ ⋆ Extras ⋆ ⋆ ⋆

Feel free to do these on the back of the previous page or elsewhere. Just tell me where to look.

A. (⋆) Use the Weierstrass substitution to transform the integral $\int \frac{2\sin x + 1}{2\sin x + \cos x}\,dx$ into an integral of a rational function. (Don't evaluate the resulting integral.)

Pretty straightforward if you remember the substitution: for $u = \tan(x/2)$, we get $dx = 2du/(1+u^2)$, $\sin x = 2u/(1+u^2)$ and $\cos x = (1-u^2)/(1+u^2)$. The integral becomes

$$\int \frac{\frac{4u}{1+u^2}+1}{\frac{4u}{1+u^2}+\frac{1-u^2}{1+u^2}}\,\frac{2du}{1+u^2} = 2\int \frac{(1+4u+u^2)du}{(1+u^2)(1+4u-u^2)}$$

B. (⋆) Calculate the arc length of the perimeter of the region bounded by the two curves $y = x^3$ and $y = x^2 - 3x$. (Set it up. Show no fear.)

The curves intersect where $x^3 = x^2 - 3x$, i.e., where $x(x^2 - x + 3) = 0$, which turns out to be only at $x = 0$, so the problem makes no sense!!! Dammit, it should have been $y = x^2 + 3x$. Sorry.

C. (⋆) The "Horn of Gabriel" is the solid of revolution obtained by rotating the curve $y = 1/x$ about the x-axis for $x > 1$, i.e., on the interval $x \in [1, \infty)$. Show that the volume of this solid is finite but that its surface area is infinite (calculate both). (This "paradox" is supposed to freak you out, so don't think about the implications until later.)

You can find this in most texts.

D. ($\star$) You'd "set up" a partial fraction decomposition for the function $\dfrac{1}{x(x+2)}$ by writing

$$\frac{1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$$

and solving for the unknown constants A and B. How would you set up the partial fraction decomposition of the following function?

$$\frac{x^5 - x^4}{(x+1)^3(x^2+2)^2}$$

No long division is necessary, since the degree of the numerator (5) is less than that of the denominator (7). So just mind your repeated fractions and write,

$$\frac{x^5 - x^4}{(x+1)^3(x^2+2)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{Dx+E}{x^2+2} + \frac{Fx+G}{(x^2+2)^2}.$$

It turns out that the actual decomposition is

$$-\frac{16}{27(x+1)} + \frac{19}{27(x+1)^2} - \frac{2}{9(x+1)^3} + \frac{8(2x-1)}{27\,(x^2+2)} - \frac{4(4x-7)}{27\,(x^2+2)^2}$$

E. ($\star$) The standard procedure for decomposing the function $\dfrac{x^3+2}{(x-1)(x+2)}$ using partial fractions would be to first do a long division and to then decompose the "remainder" using methods alluded to above. But it is clear that the result will be that

$$\frac{x^3+2}{(x-1)(x+2)} = Ax + B + \frac{C}{x-1} + \frac{D}{x+2},$$

for some constants A, B, C, D. Instead of using long division, use the tricks that we have typically applied to the remainder term to solve for these constants. (*Start* by multiplying in the usual way, to clear the denominators, then choosing convenient values for x ...)

More typos! But surely you see what was meant. Multiplying by the denominators gives

$$x^3 + 2 = (Ax+B)(x-1)(x+2) + C(x-1) + D(x+2).$$

Letting $x = 1$ gives $1^3 + 2 = D(1+2)$, so $D = 1$. Letting $x = -2$ gives $(-2)^3 + 2 = C(-2-1)$, so $C = 2$. Equating powers of x^3 shows at once that $A = 1$. There are many ways to get B, including equating the constant terms: let $x = 0$ to get $2 = -2B - C + 2D = -2B - 2 + 2$, hence $B = -1$.

F. ($\star$) Refer to problem #5e and consider the integral $\displaystyle\int_{-\infty}^{\infty} \frac{e^x\,dx}{e^{2x}+1}$, where the integrand hasn't changed but the limits of integration are now $-\infty$ to ∞. Show that

$$\int_{-\infty}^{\infty} \frac{e^x\,dx}{e^{2x}+1} = 2\int_{-\infty}^{0} \frac{e^x\,dx}{e^{2x}+1}.$$

(Can symmetry be used? Explain fully.)

Using the straightforward approach employed in #5e, let $u = e^x$ and revise the limits to get

$$\int_{-\infty}^{\infty} \frac{e^x\,dx}{e^{2x}+1} = \int_{0}^{\infty} \frac{1}{1+u^2}du = \arctan(\infty) - \arctan(0) = \pi/2.$$

Regarding symmetry, we do see that this answer is twice the result for the integral on just $(-\infty, 0]$. It turns out that you can use symmetry — although it might not appear at first glance that the integrand has even symmetry, it is revealed by writing it as follows (after dividing numerator and denominator by e^x):

$$\frac{e^x}{e^{2x}+1} = \frac{1}{e^x + e^{-x}}.$$

Hence we are integrating an even function on a symmetric interval and we can simply double our original result.

G. ($\star\star$) Stare 'em down! Without evaluating the integrals, decide which of the following are convergent (finite) and which are divergent. *No credit will be given without a short, complelling explanation.*

(a) $\displaystyle\int_{1}^{\infty} \frac{2-\sqrt{2}\cos x}{e^x}\,dx$

It's convergent. $\displaystyle\int_{1}^{\infty} \frac{1}{e^x}\,dx$ converges (and therefore any constant multiple of it converges) and

$$0 \le \frac{2-\sqrt{2}\cos x}{e^x} \le \frac{2+\sqrt{2}}{e^x}, \quad \forall x \in [1,\infty).$$

(We've also used the facts that $-1 \le \cos\theta \le 1$ for all θ and $2 > \sqrt{2}$.) The Comparison Theorem now gives convergence, since

$$0 < \int_{1}^{\infty} \frac{2-\sqrt{2}\cos x}{e^x}\,dx < \int_{1}^{\infty} \frac{2+\sqrt{2}}{e^x}\,dx < \infty.$$

(b) $\displaystyle\int_2^\infty \frac{1}{e^{2x}-x}\,dx$

At a glance, we suspect it converges — the integrand "looks like" $1/e^{2x}$, the integral of which would converge on $[2,\infty)$. But using the Comparison Theorem directly with these functions is not possible, since $\dfrac{1}{e^{2x}-x}$ is *greater* than $\dfrac{1}{e^{2x}}$ on the interval $[2,\infty)$. Mathematically speaking, the integrand *asymptotic to* the function $1/e^{2x}$, which means that the limit of the ratio of the two approaches 1:

$$\lim_{x\to\infty} \frac{1/(e^{2x}-x)}{1/e^{2x}} = 1.$$

Here is the rigorous approach to this: since the above limit is 1, we can say that the ratio in question is eventually between, say, $1/2$ and $3/2$, for sufficiently large x. (Think about what the limit actually means to understand this.) This lets us argue that, for large enough x, say $x \geq N$, we'll have

$$0 < \frac{1}{2}\left(\frac{1}{e^{2x}}\right) < \frac{1}{e^{2x}-x} < \frac{3}{2}\left(\frac{1}{e^{2x}}\right),$$

to which we can now apply the Comparison Theorem on the interval $[N,\infty)$. It is clear that the interval converges on $[2,N]$, of course, so the original improper integral is a sum of two convergent integrals and is therefore convergent.

(c) $\displaystyle\int_3^\infty \frac{x}{e^{3x}}\,dx$

"Exponentials beat polynomials," so this converges. WHAT?? Okay, to be technical, we know that $\lim_{x\to\infty} x/e^{2x} = 0$, so eventually x/e^{2x} is as small (but positive) as we like, say, $0 < x/e^{2x} < 1/2$. So we have

$$0 < \frac{x}{e^{3x}} = \frac{x}{e^{2x}} \cdot \frac{1}{e^x} < \frac{1}{2} \cdot \frac{1}{e^x},$$

the last term giving a convergent integral.

(d) $\displaystyle\int_4^\infty \frac{x^3+3x+1}{x^4+2x+2}\,dx$

This one is divergent to ∞ — it looks like $1/x$, which diverges. Note that the Comparison Theorem can actually be directly used for the two functions, but this can be avoided, the details being similar to those above. See if you can write the argument each way.

(e) $\displaystyle\int_5^\infty \frac{x^2+2x+1}{x^4+2x+2}\,dx$

Convergent – it looks like $1/x^2$. See how easy it is? As an exercise, you should supply the missing details. But the point is, you can just look at these and decide convergence or divergence, even when doing the integral is impossible or just no fun.

H. ($\star\cdots\star$) Ask a question you wish I had asked and answer it. Points may vary. Offer void where prohibited by law.