Calc II
MATH 222

Fall 2005
Name:

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# T3

**Instructions:** Write answers to problems *on separate paper.* You may NOT use calculators or any electronic devices or notes of any kind. Each st$\star$rred problem is extra credit and each $\star$ is worth 5 points. (These are just more problems, but harder. They're worth fewer points so that you're not unduly tempted.) Loads of points are possible on the test, but the highest grade that I will award is 115 points.

1. (6 points) Write the formulas for $\bar{x}$ and $\bar{y}$, where $(\bar{x}, \bar{y})$ is the centroid of a plane region lying between two curves $y = f(x)$ and $y = g(x)$ with $f(x) < g(x)$ and $a \le x \le b$.

$$\bar{x} = \frac{\int_a^b x(g(x) - f(x)\,dx}{\int_a^b (g(x) - f(x)\,dx}, \quad \bar{y} = \frac{\int_a^b \frac{1}{2}((g(x))^2 - (f(x))^2\,dx}{\int_a^b (g(x) - f(x)\,dx}$$

2. (8 points) Find the centroid of the region bounded by the graph of the curve $y = 1 - x^2$ and the *positive* $x$- and $y$-axes. Evaluate completely and simplify. For an extra 4 points, check your answer by showing explicitly that the point lies in the region. (That does *not* mean to draw a graph, although that wouldn't hurt.)

The curve intersects the positive axes at the points $(0, 1)$ and $(1, 0)$ (too easy!) so

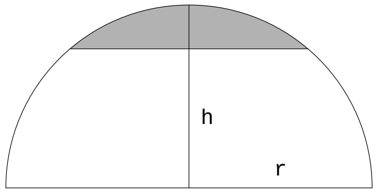
$$\bar{x} = \frac{\int_0^1 x(1 - x^2)\,dx}{\int_0^1 (1 - x^2)\,dx} = \frac{1/2 - 1/4}{1 - 1/3} = \frac{3}{8}\,.$$

Likewise,

$$\bar{y} = \frac{\frac{1}{2}\int_0^1 (1 - x^2)^2\,dx}{\int_0^1 (1 - x^2)\,dx} = \frac{1}{2} \cdot \frac{1 - 2/3 + 1/5}{1 - 1/3} = \frac{2}{5}\,.$$

Looking at the graph, it makes sense that these are both less than half, no? To check the answer for reasonableness, we can verify that the centroid lies withing the region, by verifying it lies below the curve. (This is not generally true, of course! A centroid of the form given in problem #1 need not lie below the "top" curve. Can you give an example? In our case it is true, but why?) It is easy to verify that $2/5 < 1 - (3/8)^2$, which is all that my question amounts to.

3. (8 points) Recall that we proved that the centroid of the "semidisk" of radius $r$ (the region contained in the semicircle of radius $r$) lies $\dfrac{4r}{3\pi}$ units above its base. Find the centroid of the *segment* of the disk of radius $r$ at height $h$, as shown (shaded) in the figure below. (Set it up.)




For some reason, the $h$ and $r$ on the diagram were corrupted to $i$ and $s$ in the printing. Nobody asked about, so I'll assume it was otherwise self-expolanatory.

I'll use $r = 1$ and rescale it when I'm done. I'll let the circle be the unit circle (centered at $(0,0)$. It is clear that $\bar{x} = 0$, so we only need $\bar{y}$.

If you draw line segments from the center of the disk to the points on the circle at the base of the disk segment, you'll see right triangles and you can calculate that that $x$ runs from $-\sqrt{1-h^2}$ to $\sqrt{1-h^2}$. The region is bounded by the functions $g(x) = \sqrt{1-x^2}$ and $f(x) = h$. That does it:

$$\bar{y} = \frac{\displaystyle\int_{-\sqrt{1-h^2}}^{\sqrt{1-h^2}} \frac{1}{2}((\sqrt{1-x^2})^2 - h^2)dx}{\displaystyle\int_{-\sqrt{1-h^2}}^{\sqrt{1-h^2}} (\sqrt{1-x^2} - h)dx}$$

Yes, you can use symmetry to clean it up.

4. (5 points) Express the number $2.\overline{34} = 2.34343434\cdots$ as a ratio of integers.

Slap some 9's: the answer is $2 + 34/99 = 232/99$. Use a geometric series or other trick.

5. (5 points) State the Monotonic Convergence Theorem. (I called it the "Big Man on Campus Theorem".) Look it up.

6. (8 points) Define the sequence $\{a_n\}$ as follows. Let $a_1 = \pi$ and for $n \geq 1$, define

$$a_{n+1} = \frac{1}{2}(a_n + 5).$$

Prove that the sequence is convergent.

There is an example worked in the text that is similar. Basically, each suggessive element in the sequence is the avarage of the previous one with 5. Since the sequence starts at $\pi$, we expect the numbers to increase to 5. We can prove this without finding a formula for the $n$'th term (although that isn't hard) by showing the sequence is bounded above (by 5) and increasing. We use induction first to show the sequence is bdd.

Clearly, $a_1 = \pi < 5$, so the "base case" is true. Now suppose that $a_n \leq 5$. Then

$$\begin{aligned} a_{n+1} &= \frac{1}{2}(a_n + 5) \\ &\leq \frac{1}{2}(5 + 5) \qquad \text{(by the inductive hypothesis)} \\ &= 5. \end{aligned}$$

So we have proved that $a_n \leq 5$ for all $n \geq 1$, i.e., that $\{a_n\}$ is bdd above by 5. Showing $\{a_n\}$ is increasing is now easy:

$$\begin{aligned} a_{n+1} - a_n &= \frac{1}{2}(a_n + 5) - a_n \\ &= \frac{1}{2}(5 - a_n) \\ &> \frac{1}{2}(0) \qquad \text{(since } a_n < 5\text{)} \\ &= 0, \end{aligned}$$

hence $a_{n+1} \geq a_n$. The result is proved.

7. (5 points) Write the formula for the sum of the geometric series, $\sum_{k=p}^{\infty} c^k$, where $-1 < c < 1$.

$$\frac{c^p}{1-c}$$

8. (2 points each) Determine whether each of the following **sequences** $\{a_n\}$ converges or diverges for the given $a_n$. If it converges, find its limit. If it diverges to $+\infty$, say so. If it diverges to $-\infty$, say so. If it diverges in some other way, say how. No credit for "diverges" or "converges", but no penalties for incorrect answers. If you can prove your results fairly rigorously, save that for extra credit problem E, but write it on a separate page.

(a) $a_n = \dfrac{n(n+2)}{n+1}$ div to $+\infty$

(b) $a_n = \dfrac{(-2)^{2n}}{5^n}$ conv to 0

(c) $a_n = \dfrac{2}{\ln(n+1)}$ conv to 0

(d) $a_n = \arctan\left(\dfrac{n}{n+1}\right)$ conv to $\pi/4$

(e) $a_n = \dfrac{2n^2}{(n+1)\sqrt{n^2+1}}$ conv to 2

(f) $a_n = \dfrac{n!}{(2n+1)!}$ conv to 0

(g) $a_n = \cos\dfrac{(-1)^n}{n}$ conv to 1

(h) $a_n = \dfrac{n^3}{3^n}$ conv to 0

(i) $a_n = \dfrac{n^{3n}}{3^n}$ div to $\infty$

(j) $a_n = \dfrac{n^{3n}}{3^{n^2}}$ conv to 0

9. (2 points for each correct answer, $-1$ for each incorrect answer, no penalty for blanks) Determine whether each of the following sequences is eventually increasing, eventually decreasing or not eventually monotonic. If you can prove your results fairly rigorously, save that for extra credit problem F on a separate page.

(a) $a_n = \dfrac{1}{e^n+1}$ dec

(b) $a_n = \dfrac{n}{e^n}$ dec

(c) $a_n = \dfrac{2}{3n+1}$ dec

(d) $a_n = \dfrac{n+2}{3n+1}$ dec

(e) $a_n = \dfrac{(-2)^{2n}}{5^n}$ dec

(f) $a_n = n + \dfrac{1}{n}$ inc

(g) $a_n = \arctan\left(\dfrac{n}{n+1}\right)$ inc

(h) $a_n = \dfrac{n!}{(2n+1)!}$ dec

(i) $a_n = \cos\dfrac{(-1)^n}{n}$ inc

(j) $a_n = \dfrac{n+2}{3n+1}$ dec

(k) $a_n = \dfrac{n+2}{\ln(n+1)}$ ev inc

(l) $a_n = \dfrac{\sqrt{3n+2}-\sqrt{n}}{n+\sqrt{n}}$ ev dec

10. (4 points each) Evaluate the sum of each of the following series (all are convergent). Be mindful of the lower limits in the sums.

(a) $\displaystyle\sum_{n=3}^{\infty} \frac{5}{4^n}$ $= 5/48$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2+2n}$ $= 5/12$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{3^{n-1}}$ $= -3/4$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(2^n+3^n)^2}{7^n}$ OOPS! Doesn't converge!

11. (5 points) State the hypotheses and conclusion of the theorem we call "the Integral Test" for convergence of series. Look it up.

12. (5 points) Use the integral test to decide the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n\,(\ln n)\,(\ln\ln n)}.$$

(First of all, the lower limit was meant to be 2.) This was done in class: the function $f(x) = 1/(x(\ln x)(\ln\ln x))$ is decreasing to 0; let $u = \ln\ln x$ ... the integral diverges, so the series diverges.

13. (5 points) State the hypotheses and conclusion of the theorem we call "the Limit Comparison Test" for convergence of series. Look it up.

14. (5 points) Use the limit comparison test to decide the convergence or divergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{(n-1)\sqrt{n^2+1}}.$$

Compare it with $\sum 1/n^2$, which converges.

15. (5 points) State the hypotheses and conclusion of the theorem we call "the Alternating Series Test" for convergence of series. Look it up.

16. (1 point for each correct answer, $-1$ for each incorrect answer, no penalty for blanks) Determine whether each of the following **series** is convergent or divergent. If you can justify your results with reasonable clarity and brevity, save that for extra credit problem J on a separate page.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n+1}$ conv

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$ conv

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3n+1}$ div

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(2-n)^2}{3n^2+1}$ div

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{3n}}{3^{2n}}$ conv

(f) $\displaystyle\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n+1}\right)$ div

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$ conv

(h) $\displaystyle\sum_{n=1}^{\infty} \sin\frac{(-1)^n}{n}$ conv

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n+\sin n}{n^3+2n}$ conv

(j) $\displaystyle\sum_{n=1}^{\infty} \arcsin\frac{3n+1}{2n^3+n}$ conv

(k) $\displaystyle\sum_{n=1}^{\infty} \cos\frac{(-1)^n(n+1)}{2n^2+3}$ div

(l) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ conv

(m) $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^2}$ conv

(n) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ln n}{n}$ conv

(o) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ conv

(p) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^n}{n!}$ div

## $\star \quad \star \quad \star$ Extras $\star \quad \star \quad \star$

Feel free to do these on the back of the previous page or elsewhere. Just tell me where to look.

A. ($\star$) Write the approximation to the area under the curve $y = \sqrt{2 - x^3}$ between $x = 0$ and $x = 1$ by applying the trapezoid rule with 5 equal subintervals.

B. ($\star$) State and prove the theorem of Pappus.

C. ($\star$) Find the volume of the solid of revolution obtained by rotating the region described in problem #1 about

(a) the $x$-axis;

(b) the $y$-axis;

(c) the line $x = 4$;

(d) the line $y = -2$.

D. ($\star\star$) Same as the previous problem, but about the line

(a) the line $x = 1/2$;

(b) the line $y = x$.

E. ($\star\star$) For up to two points each, prove five of your answers in problem #8.

F. ($\star\star$) For up to two points each, prove five of your answers in problem #9.

G. ($\star\star$) Discuss the convergence or divergence of these sequences.

(a) $\left\{\sqrt[n]{3^n + 5^n}\right\}$.

(b) $\left\{\dfrac{\ln 5n}{\ln 3n}\right\}$

(c) $\left\{\left(1 + \dfrac{5}{n}\right)^{3n}\right\}$

H. ($\star$) Find the value of the following repeating continued fraction.

$$[3; 2, 3, 2, 3, 2, 3, \ldots] = 3 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \ddots}}}$$

I. ($\star$) We write our numbers in base-10, but what if we were members of a race with 6 digits on each hand? Aside from the musical implications, we'd probably be using a base-12 number system. Find the exact fraction, expressed as a rational number (i.e., a ratio of integers), that is represented by the base-12 number

$$0.\mathrm{ABBBBBBB}\cdots,$$

where A and B are is the base-12 digits for ten and eleven, respectively.

J. ($\star\star\star$) For up to 1 point each, briefly justify your answers in problem #16.

K. ($\star\star$) Discuss the convergence or divergence of these series.

(a) $\displaystyle\sum_{n=1}^{\infty} \sin \frac{n^2\pi + 1}{n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$

L. ($\star$) Use mathematical induction to prove that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

M. ($\star \cdots \star$) Ask a question you wish I had asked and answer it. Points may vary. Offer void where prohibited by law.