Calc II | Fall 2005

MATH 222 | Name: ____________

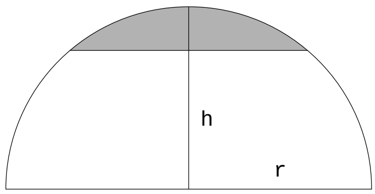
T3

Instructions: Write answers to problems *on separate paper.* You may NOT use calculators or any electronic devices or notes of any kind. Each st⋆rred problem is extra credit and each ⋆ is worth 5 points. (These are just more problems, but harder. They're worth fewer points so that you're not unduly tempted.) Loads of points are possible on the test, but the highest grade that I will award is 115 points.

1. (6 points) Write the formulas for $\bar{x}$ and $\bar{y}$, where $(\bar{x}, \bar{y})$ is the centroid of a plane region lying between two curves $y = f(x)$ and $y = g(x)$ with $f(x) < g(x)$ and $a \leq x \leq b$.

2. (8 points) Find the centroid of the region bounded by the graph of the curve $y = 1 - x^2$ and the *positive* x- and y-axes. Evaluate completely and simplify. For an extra 4 points, check your answer by showing explicitly that the point lies in the region. (That does *not* mean to draw a graph, although that wouldn't hurt.)

3. (8 points) Recall that we proved that the centroid of the "semidisk" of radius r (the region contained in the semicircle of radius r) lies $\dfrac{4r}{3\pi}$ units above its base. Find the centroid of the *segment* of the disk of radius r at height h, as shown (shaded) in the figure below. (Set it up.)



4. (5 points) Express the number $2.\overline{34} = 2.34343434\cdots$ as a ratio of integers.

5. (5 points) State the Monotonic Convergence Theorem. (I called it the "Big Man on Campus Theorem".)

6. (8 points) Define the sequence $\{a_n\}$ as follows. Let $a_1 = \pi$ and for $n \geq 1$, define

$$a_{n+1} = \frac{1}{2}(a_n + 5).$$

Prove that the sequence is convergent.

7. (5 points) Write the formula for the sum of the geometric series, $\sum_{k=p}^{\infty} c^k$, where $-1 < c < 1$.

8. (2 points each) Determine whether each of the following **sequences** $\{a_n\}$ converges or diverges for the given a_n. If it converges, find its limit. If it diverges to $+\infty$, say so. If it diverges to $-\infty$, say so. If it diverges in some other way, say how. No credit for "diverges" or "converges", but no penalties for incorrect answers. If you can prove your results fairly rigorously, save that for extra credit problem E, but write it on a separate page.

   (a) $a_n = \dfrac{n(n+2)}{n+1}$

   (b) $a_n = \dfrac{(-2)^{2n}}{5^n}$

   (c) $a_n = \dfrac{2}{\ln(n+1)}$

   (d) $a_n = \arctan\left(\dfrac{n}{n+1}\right)$

   (e) $a_n = \dfrac{2n^2}{(n+1)\sqrt{n^2+1}}$

   (f) $a_n = \dfrac{n!}{(2n+1)!}$

   (g) $a_n = \cos\dfrac{(-1)^n}{n}$

   (h) $a_n = \dfrac{n^3}{3^n}$

   (i) $a_n = \dfrac{n^{3n}}{3^n}$

   (j) $a_n = \dfrac{n^{3n}}{3^{n^2}}$

9. (2 points for each correct answer, -1 for each incorrect answer, no penalty for blanks) Determine whether each of the following sequences is eventually increasing, eventually decreasing or not eventually monotonic. If you can prove your results fairly rigorously, save that for extra credit problem F on a separate page.

   (a) $a_n = \dfrac{1}{e^n+1}$

   (b) $a_n = \dfrac{n}{e^n}$

   (c) $a_n = \dfrac{2}{3n+1}$

   (d) $a_n = \dfrac{n+2}{3n+1}$

   (e) $a_n = \dfrac{(-2)^{2n}}{5^n}$

   (f) $a_n = n + \dfrac{1}{n}$

   (g) $a_n = \arctan\left(\dfrac{n}{n+1}\right)$

   (h) $a_n = \dfrac{n!}{(2n+1)!}$

   (i) $a_n = \cos\dfrac{(-1)^n}{n}$

   (j) $a_n = \dfrac{n+2}{3n+1}$

   (k) $a_n = \dfrac{n+2}{\ln(n+1)}$

   (l) $a_n = \dfrac{\sqrt{3n+2}-\sqrt{n}}{n+\sqrt{n}}$

10. (4 points each) Evaluate the sum of each of the following series (all are convergent). Be mindful of the lower limits in the sums.

   (a) $\displaystyle\sum_{n=3}^{\infty} \frac{5}{4^n}$

   (b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2+2n}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{3^{n-1}}$

   (d) $\displaystyle\sum_{n=0}^{\infty} \frac{(2^n+3^n)^2}{7^n}$

11. (5 points) State the hypotheses and conclusion of the theorem we call "the Integral Test" for convergence of series.

12. (5 points) Use the integral test to decide the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n\,(\ln n)\,(\ln \ln n)}.$$

13. (5 points) State the hypotheses and conclusion of the theorem we call "the Limit Comparison Test" for convergence of series.

14. (5 points) Use the limit comparison test to decide the convergence or divergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{(n-1)\sqrt{n^2+1}}.$$

15. (5 points) State the hypotheses and conclusion of the theorem we call "the Alternating Series Test" for convergence of series.

16. (1 point for each correct answer, -1 for each incorrect answer, no penalty for blanks) Determine whether each of the following **series** is convergent or divergent. If you can justify your results with reasonable clarity and brevity, save that for extra credit problem J on a separate page.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n+1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3n+1}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(2-n)^2}{3n^2+1}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{3n}}{3^{2n}}$

(f) $\displaystyle\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n+1}\right)$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$

(h) $\displaystyle\sum_{n=1}^{\infty} \sin\frac{(-1)^n}{n}$

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n+\sin n}{n^3+2n}$

(j) $\displaystyle\sum_{n=1}^{\infty} \arcsin\frac{3n+1}{2n^3+n}$

(k) $\displaystyle\sum_{n=1}^{\infty} \cos\frac{(-1)^n(n+1)}{2n^2+3}$

(l) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

(m) $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^2}$

(n) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ln n}{n}$

(o) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

(p) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^n}{n!}$

$\star \quad \star \quad \star$ Extras $\star \quad \star \quad \star$

Feel free to do these on the back of the previous page or elsewhere. Just tell me where to look.

A. ($\star$) Write the approximation to the area under the curve $y = \sqrt{2 - x^3}$ between $x = 0$ and $x = 1$ by applying the trapezoid rule with 5 equal subintervals.

B. ($\star$) State and prove the theorem of Pappus.

C. ($\star$) Find the volume of the solid of revolution obtained by rotating the region described in problem #1 about

(a) the x-axis;

(b) the y-axis;

(c) the line $x = 4$;

(d) the line $y = -2$.

D. ($\star\star$) Same as the previous problem, but about the line

(a) the line $x = 1/2$;

(b) the line $y = x$.

E. ($\star\star$) For up to two points each, prove five of your answers in problem #8.

F. ($\star\star$) For up to two points each, prove five of your answers in problem #9.

G. ($\star\star$) Discuss the convergence or divergence of these sequences.

(a) $\left\{\sqrt[n]{3^n + 5^n}\right\}$.

(b) $\left\{\dfrac{\ln 5n}{\ln 3n}\right\}$

(c) $\left\{\left(1 + \dfrac{5}{n}\right)^{3n}\right\}$

H. ($\star$) Find the value of the following repeating continued fraction.

$$[3; 2, 3, 2, 3, 2, 3, \ldots] = 3 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \ddots}}}$$

I. ($\star$) We write our numbers in base-10, but what if we were members of a race with 6 digits on each hand? Aside from the musical implications, we'd probably be using a base-12 number system. Find the exact fraction, expressed as a rational number (i.e., a ratio of integers), that is represented by the base-12 number

$$0.\text{ABBBBBBB}\cdots,$$

where A and B are is the base-12 digits for ten and eleven, respectively.

J. ($\star\star\star$) For up to 1 point each, briefly justify your answers in problem #16.

K. ($\star\star$) Discuss the convergence or divergence of these series.

(a) $\displaystyle\sum_{n=1}^{\infty} \sin \frac{n^2\pi + 1}{n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$

L. ($\star$) Use mathematical induction to prove that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

M. ($\star\cdots\star$) Ask a question you wish I had asked and answer it. Points may vary. Offer void where prohibited by law.