Calc II
MATH 222

Spring 2009
Name: ____________________

T3

Instructions: Write answers to problems *on separate paper* and be sure everything you turn in is in order. You may NOT use calculators or any electronic devices or notes of any kind. Each st⋆rred problem is extra credit and each ⋆ is worth 4 points. (These are just more problems, but harder. They're worth fewer points so that you're not unduly tempted.) Loads of points are possible on the test, but the highest grade that I will award is 110 points.

1. (2 points each) Determine whether each of the following **sequences** $\{a_n\}$ converges or diverges for the given a_n. (Warning: do not confuse "sequence" with "series".) If the sequence converges, find its limit. If it diverges to $+\infty$, say so. If it diverges to $-\infty$, say so. If it diverges in some other way, say how. No credit for "diverges" or "converges", but no penalties for incorrect answers. No explanations wanted here, just do like a ninja on these.

   (a) $a_n = \dfrac{2n+3n^2}{n^2+1}$

   (b) $a_n = (-1)^n \dfrac{2n+3n^2}{n^2+1}$

   (c) $a_n = (-1)^n \left(\dfrac{2n^3+3n}{n^2+1}\right)$

   (d) $a_n = \dfrac{\ln(n^3)}{(3n)^2}$

   (e) $a_n = \dfrac{1}{\ln(n^2+1)}$

   (f) $a_n = \sin\left(\dfrac{(-1)^n}{n^2}\right)$

   (g) $a_n = \dfrac{(2n)!}{2(n+1)!}$

   (h) $a_n = \dfrac{(3n+1)\sqrt{5n^4+n}}{2n^3+1}$

   (i) $a_n = n\sin\dfrac{1}{n}$

   (j) $a_n = \dfrac{5^n}{5^n+2^n}$

2. (3 points) State the definition of a convergent sequence. (Hint: It has an epsilon in it.)

3. (3 points) State *precisely* what it means to say that $\displaystyle\sum_{n=1}^{\infty} a_n = L$.

4. (3 points) State (the hypotheses and conclusions of) the monotone convergence theorem.

5. (3 points) State the precise hypotheses and conclusion of the theorem we call "the Limit Comparison Test".

6. (3 points) State the precise hypotheses and conclusion of the theorem we call "the Integral Test".

7. (3 points) State the precise hypotheses and conclusion of the theorem we call "the Root Test".

8. (2 points for each correct answer, -1 for each incorrect answer, no penalty for blanks) Determine whether each of the following series is **abs**olutely convergent, **cond**itionally convergent or **div**ergent. Do not give reasons for your answers, just ninja-fy these.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{2n^2}{3n^3+1}$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{n!}$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{n^3}{2^n}$

(d) $\displaystyle\sum_{n=1}^{\infty}\cos\frac{1}{n^2}$

(e) $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$

(f) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[n]{3}}$

(g) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{2^{3n}}{3^{2n}}$

(h) $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n+1}(\ln n)^3}{n}$

(i) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\cos n}{(n+1)^2}$

(j) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\arctan(1/n)$

(k) $\displaystyle\sum_{n=1}^{\infty}(-1)^n/\arctan(n)$

9. (5 points each) Determine whether each of the following series is absolutely convergent, conditionally convergent or divergent, and explain your answers carefully but succinctly. The clarity of your reasoning is what gets points on these.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n^3}{2^n}$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1/3}}$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1/n}}$

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n 2^n}{n^n}$

10. (5 points each) Write the *intervals* of convergence of each of the power series. (I.e., state all real values of x for which the series converges. Don't ignore the endpoints, if any.)

(a) $\displaystyle\sum_{n=1}^{\infty}n^2(3x)^n$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{3n^2}$

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{(2x)^n}{n!}$

(d) $\displaystyle\sum_{n=0}^{\infty}n!(x-2/3)^n$

11. (5 points each) For each function $f(x)$ below, write a power series (either in summation notation or in "pattern $\cdots$ form" and including the first four terms) centered about the point $x=c$. Then give the interval of convergence of the series.

(a) $f(x)=\frac{2}{3+4x}$; $c=0$;

(b) $f(x)=\frac{4}{3-2x}$; $c=5$;

12. (5 points each) Find the exact sum of each series. Be careful of the lower index in the sum.

(a) $\displaystyle\sum_{n=3}^{\infty}\frac{1}{2^n}$

(b) $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n(n-1)}$

(c) $\displaystyle\sum_{n=3}^{\infty}\frac{n}{2^n}$ (Hint: a diff?)

$\star \quad \star \quad \star \quad \star \quad \star$ Extras $\star \quad \star \quad \star \quad \star \quad \star$

A. ($\star$) Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

B. ($\star$) State Stirling's formula.

C. ($\star$) The sequence $\left\{\dfrac{n!}{2^n}\right\}$ diverges. Prove that the sequence $\left\{\dfrac{n!}{2^{n^2}}\right\}$ converges. (This implies that 2^{n^2} seriously whomps $n!$ *!*)

D. ($\star\star$) Find the exact sum of each series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2}{n!}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,2^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$

E. ($\star$) Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{|n^2-3n|}}{2n^2+1}$. Its first few terms are approximately as follows.

$$\{-0.471405,\ 0.157135,\ 0,\ 0.0606061,\ -0.0620054,\ \dots\}$$

(a) Explain why it is immediately clear (without resorting to calculations) whether the series converges or diverges (and say which).

(b) It turns out that

$$\frac{d}{dx}\left(\frac{\sqrt{x^2-3x}}{2x^2+1}\right) = -\frac{4x^3-18x^2-2x+3}{2\sqrt{x^2-3x}\,(2x^2+1)^2}.$$

Use this to confirm your answer in part (a).

F. ($\star\cdots\star$) Ask a question you wish I had asked and answer it. Points may vary. Offer void where prohibited by law.