Calc III — Fall 2002

MATH 223 — Name: ____________________

# *Final Exam*

A.) **The problems in this section are worth 6 points each. A maximum of 60 points will be awarded in this section.**

1. Consider the triangle having vertices $P = (2, 1, 1)$, $Q = (-1, 4, 2)$ and $R = (0, -2, 1)$.

   a.) Find the area of $\triangle PQR$

   Use the cross-product:

$$\begin{aligned} |\triangle PQR| &= \frac{1}{2}\,||\mathbf{PQ} \times \mathbf{PR}|| \\ &= \frac{1}{2}\,||\langle -3, 3, 1\rangle \times \langle -2, -3, 0\rangle|| \\ &= \frac{1}{2}\,||\langle 3, -2, 15\rangle|| = \frac{1}{2}\sqrt{238} \end{aligned}$$

   b.) Let $Q'$ denote the point that lies $2/3$ the way from $P$ to $Q$; let $R'$ be the point lying $3/4$ the way from $P$ to $R$. Find the coördinates of $Q'$ and $R'$.

$$\begin{aligned} Q' &= P + (2/3)\mathbf{PQ} = (2, 1, 1) + (2/3)\langle -3, 3, 1\rangle \\ &= (2, 1, 1) + \langle -2, 2, 2/3\rangle = (0, 3, 5/3), \end{aligned}$$

$$\begin{aligned} R' &= P + (3/4)\mathbf{PR} = (2, 1, 1) + (3/4)\langle -2, -3, 0\rangle \\ &= (2, 1, 1) + \langle -3/2, -9/4, 0\rangle = (1/2, -5/4, 1). \end{aligned}$$

   c.) Find the area of $\triangle PQ'R'$

   One way is to just haul off and do what you did in part (a):

$$\begin{aligned} |\triangle PQ'R'| &= \frac{1}{2}\,||\mathbf{PQ'} \times \mathbf{PR'}|| \\ &= \frac{1}{2}\,||\langle -2, 2, 2/3\rangle \times \langle -3/2, -9/4, 0\rangle|| \\ &= \frac{1}{2}\,||\langle -135/8, 45/4, 39/8\rangle|| = \frac{1}{2}\sqrt{119/2} = \frac{1}{4}\sqrt{238}. \end{aligned}$$

   Or you could simply note that

$$\begin{aligned} |\triangle PQ'R'| &= \left(\frac{2}{3}\right)\left(\frac{3}{4}\right)|\triangle PQR| \\ &= \frac{1}{2}|\triangle PQR| = \frac{1}{4}\sqrt{238}. \end{aligned}$$

2. At what point does the line

$$L = \{(2t-1, t+3, 1-t) \mid t \in \mathbb{R}\}$$

meet the plane containing $\triangle PQR$ of problem #A1?

First get an equation for the plane. We already have a normal vector from problem #A1:

$$\mathbf{PQ} \times \mathbf{PR} = \langle 3, -2, 15 \rangle.$$

So an equation for the plane is

$$\langle 3, -2, 15 \rangle \cdot \langle x, y, z \rangle = \langle 3, -2, 15 \rangle \cdot P,$$

or

$$3x - 2y + 15z = 19.$$

Now just plug in a point on $L$ and solve for $t$:

$$3(2t-1) - 2(t+3) + 15(1-t) = 19$$

$$\Rightarrow t = -13/11.$$

The point is therefore

$$(2(-13/11) - 1, -13/11 + 3, 1 - (-13/11)) = (-37/11, 20/11, 24/11).$$

3. Find the distance from the line $L$ of problem #A2 to the point $P$ of problem #A1.

The distance from a line $T + t\mathbf{v}$ to a point $P$ is $\dfrac{||\mathbf{PT} \times \mathbf{v}||}{||\mathbf{v}||}$. We can express $L$ with

$$T = (-1, 3, 1), \quad \mathbf{v} = \langle 2, 1, -1 \rangle,$$

so our distance is

$$\frac{||\langle -3, 2, 0 \rangle \times \langle 2, 1, -1 \rangle||}{||\langle 2, 1, -1 \rangle||} = \frac{||\langle -2, -3, -7 \rangle||}{||\langle 2, 1, -1 \rangle||} = \frac{\sqrt{62}}{\sqrt{6}}$$

4. Referring again to $\triangle PQR$ of problem #A1, a perpendicular is dropped from point $Q$ to a point $S$ on the line containing segment $PR$. Find the coordinates of $S$.

Since $S$ is the projection of $Q$ onto $\mathbf{PR}$, we have

$$\begin{aligned}
S &= P + \frac{\mathbf{PQ} \cdot \mathbf{PR}}{||\mathbf{PR}||^2}\,\mathbf{PR} \\
&= (2, 1, 1) + \frac{\langle -3, 3, 1 \rangle \cdot \langle -2, -3, 0 \rangle}{||\langle -2, -3, 0 \rangle||^2}\,\langle -2, -3, 0 \rangle \\
&= (2, 1, 1) + \frac{-3}{13}\langle -2, -3, 0 \rangle \\
&= \langle 32/13, 22/13, 1 \rangle.
\end{aligned}$$

5. Find a unit vector that bisects the angle at vertex $Q$ of the triangle $PQR$ in problem #A1.

A familiar tale. A vector bisecting the angle is found via symmetry:

$$||\mathbf{QP}||\,\mathbf{QR} + ||\mathbf{QR}||\,\mathbf{QP} = \sqrt{19}\langle 1 + 3\sqrt{2}, -6 - 3\sqrt{2}, -1 - \sqrt{2}\rangle.$$

Now we just need to scale this to be a unit vector. Ugly, perhaps, but just write it:

$$\frac{\langle 1 + 3\sqrt{2}, -6 - 3\sqrt{2}, -1 - \sqrt{2}\rangle}{\sqrt{\left(-1-\sqrt{2}\right)^2 + 9\left(2+\sqrt{2}\right)^2 + \left(1+3\sqrt{2}\right)^2}}\,.$$

It simplifies to

$$\frac{\langle 1 + 3\sqrt{2}, -6 - 3\sqrt{2}, -1 - \sqrt{2}\rangle}{\sqrt{76 + 44\sqrt{2}}}\,.$$

6. What kind of surface (e.g., elliptic paraboloid, hyperboloid of two sheets, etc.) is each of the following in $\mathbb{R}^3$?

   a.) $(x-1)^2 - (y+2)^2 = 3 - z$
   This is a hyperbolic paraboloid (saddle). Cross-sections perpendicular to the $z$-axis are hyperbolas (opening differently depending on the sign of $3 - z$), cross-sections perpendicular to the $x$- and $y$-axes are parabolas.

   b.) $(x-1)^2 - (y+2)^2 = z^2$
   A right circular cone. Add the negative term to the other side to see it. It has a vertex at $(1, -2, 0)$ and its axis is parallel to the $x$-axis.

   c.) $2x^2 + 3y = 5$
   This is a parabolic cyliner. The $z$ coördinate is free; for each $z$ we get a parabola.

7. Let $\mathbf{R}(t) = \langle 2t^2, 2t - t^2, 1 - t\rangle$ denote a space curve in $\mathbb{R}^3$ (it's a parabola). Find its minimum radius of curvature.

The curvature is

$$\frac{||\mathbf{R}'(t) \times \mathbf{R}''(t)||}{||\mathbf{R}'(t)||^3},$$

so the radius of curvature is the reciprocal of that, which is

$$r(t) = \frac{(20t^2 - 8t + 5)^{3/2}}{\sqrt{84}}.$$

This is minimum when $20t^2 - 8t + 5$ is minimum, which is when $t = 1/5$ (just take the derivative and set it equal to zero, or note that the vertex of a parabola, which is at the minimum here, is where $t = -b/2a$). The actual minimum of the quadratic is

$$20(1/5)^2 - 8(1/5) + 5 = 21/5,$$

so the minimum radius is

$$(21/5)^{3/2}/\sqrt{84}.$$

8. Suppose that the acceleration of a particle is described by

$$\mathbf{A}(t) = \langle t-2, 4, 6\sin \pi t\rangle,$$

for each $t \in \mathbb{R}$. The particle is observed to have a velocity vector

$$\mathbf{V_0} = \langle 0, 2, 0\rangle$$

when it is at the point $(-2, 4, 0)$. Find a formula for the position vector as a function of $t$.
Integrate once to get the velocity vector.

$$\mathbf{V}(t) = \int \mathbf{A}(t)\, dt = \int \langle t-2, 4, 6\sin \pi t\rangle\, dt = \langle t^2/2 - 2t, 4t, -(6/\pi)\cos \pi t\rangle + \mathbf{C}_1.$$

When the particle is at $(-2, 4, 0)$, we see that $t = 0$ (you're welcome); we know that the velocity is $\langle 0, 2, 0\rangle$ there, so we can find $\mathbf{C}_1$:

$$\langle 0, 0, -6/\pi\rangle + \mathbf{C}_1 = \langle 0, 2, 0\rangle,$$

which implies that $\mathbf{C}_1 = \langle 0, 2, 6/\pi\rangle$. Integrating again, we have the position vector:

$$\begin{aligned}\mathbf{R}(t) &= \int \mathbf{V}(t)\, dt = \int (\langle t^2/2 - 2t, 4t, -(6/\pi)\cos \pi t\rangle + \mathbf{C}_1)dt \\ &= \langle t^3/6 - t^2, 2t^2, -6/\pi^2 \sin \pi t\rangle + \mathbf{C}_1 t + \mathbf{C}_2.\end{aligned}$$

At $t = 0$ this gives

$$\mathbf{C}_2 = \langle -2, 4, 0\rangle.$$

9. What is the curvature of the path of the particle in problem #A8, when $t = 0$?
Even without doing problem #A8, the curvature is given by

$$\frac{||\mathbf{R}'(0) \times \mathbf{R}''(0)||}{||\mathbf{R}'(0)||^3} = \frac{||\langle 0, 2, 0\rangle \times \langle -2, 4, 0\rangle||}{||\langle 0, 2, 0\rangle||^3} = 4/2^{3/2} = \sqrt{2}.$$

10. For each of the following, determine whether or not $f(x, y)$ has a limit at $(0, 0)$. *Justify your answers — no one-word answers.*

a.) $f(x, y) = \dfrac{x^2}{x^2 + 2y^2}$
No limit. Along the $y$-axis ($x = 0$) the limit is 0, but along the $x$-axis it's 1.

b.) $f(x, y) = \dfrac{x^3}{x^2 + 2y^2}$
The limit is zero:

$$\left|\frac{x^3}{x^2 + 2y^2}\right| = \frac{|x|x^2}{x^2 + y^2} \le |x|,$$

which goes to zero.

c.) $f(x, y) = \dfrac{x^4 y^2}{x^8 + y^4}$
No limit. Along the $y$-axis it is clearly zero, and if you check all straight lines $y = mx$ it goes to zero along these, too. But if you approach the origin along the curve $y = x^2$ you get 1/2.

11. Prove that the following function is continuous at $(0,0)$.

$$f(x,y)=\begin{cases}\dfrac{x^4-2y^4}{x^2+2y^2} & \text{if} \quad (x,y)\neq(0,0)\\[2ex] 0 & \text{if} \quad (x,y)=(0,0)\end{cases}$$

Since $f(0,0)=0$, we just need to show that the limit is zero as we approach $(0,0)$. But for $(x,y)\neq(0,0)$ we have

$$\begin{aligned}|f(x,y)| &= \left|\frac{x^4-2y^4}{x^2+2y^2}\right| \\ &\leq \frac{x^4+2y^4}{x^2+2y^2} \quad \text{(triangle inequality)} \\ &= \frac{x^2x^2+2y^2y^2}{x^2+2y^2} \\ &\leq \frac{x^2(x^2+2y^2)+(x^2+2y^2)y^2}{x^2+2y^2} \\ &= x^2+y^2,\end{aligned}$$

which goes to zero.

12. Find $f_{xy}$ for the function

$$f(x,y)=x^3+e^{3x^2y}+y^3.$$

We have

$$f_x(x,y)=3\,x^2+6\,e^{3\,x^2\,y}\,x\,y,$$

hence

$$f_{xy}(x,y)=\frac{\partial}{\partial y}f_x(x,y)=6\,e^{3\,x^2\,y}\,x+18\,e^{3\,x^2\,y}\,x^3\,y=6\,e^{3\,x^2\,y}\,x\,\left(1+3\,x^2\,y\right).$$

B.) **The problems in this section are worth 10 points each. A maximum of 60 points will be awarded in this section.**

1. Give a unit vector in the direction of greatest increase of the function $f(x,y)$ given in problem #A12.

For each $(x,y)$, the direction of greatest increase is given by the gradient at each point,

$$\begin{aligned}\nabla f(x,y) &= f_x(x,y)\mathbf{i}+f_y(x,y)\mathbf{j} \\ &= (3\,x^2+6\,e^{3\,x^2\,y}\,x\,y)\mathbf{i}+(3\,e^{3\,x^2\,y}\,x^2+3\,y^2)\mathbf{j}.\end{aligned}$$

Okay, a unit vector, which looks hideous, is just

$$\frac{(3\,x^2+6\,e^{3\,x^2\,y}\,x\,y)\mathbf{i}+(3\,e^{3\,x^2\,y}\,x^2+3\,y^2)\mathbf{j}}{\sqrt{(3\,x^2+6\,e^{3\,x^2\,y}\,x\,y)^2+(3\,e^{3\,x^2\,y}\,x^2+3\,y^2)^2}}.$$

2. Find the directional derivative of the function $f(x, y)$ given in problem #A12, in the direction of $\langle 3, -4\rangle$, at the point $(1, 1)$.

   The directional derivative at $(a, b)$ in the direction of a unit vector $\hat{\mathbf{u}}$ is just

$$D_{\hat{\mathbf{u}}} f(a, b) = \nabla f(a, b) \cdot \hat{\mathbf{u}}.$$

   Ours is

$$\nabla f(1, 1) \cdot \langle 3/5, -4/5\rangle = \langle 3 + 6e^3, 3 + 3e^3\rangle \cdot \langle 3/5, -4/5\rangle = 3(-1 + 2e^3)/5.$$

3. Find an equation for the plane tangent to the first surface given in problem #A6, at the point $(3, 0, 3)$.

   The equation is $(x - 1)^2 - (y + 2)^2 = 3 - z$ and we duly note that $(3, 0, 3)$ truly does lie on this surface. In this instance it is easiest to simply let

$$f(x, y) = z = 3 - (x - 1)^2 + (y + 2)^2,$$

   then use the formula for the tangent plane at $(a, b)$,

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

   Since

$$f_x(x, y) = -2(x - 1) \text{ and } f_y(x, y) = 2(y + 2),$$

   so that

$$f_x(3, 0) = -4 \text{ and } f_y(3, 0) = 4,$$

   we have

$$z - 3 = -4(x - 3) + 4(y - 0),$$

   or

$$z = -4x + 4y + 15.$$

4. Find all critical points of the function

$$f(x,y) = x^3 + y^3 + 3x^2 - 18y^2 + 81y + 5$$

and classify each as either a relative maximum, a relative minimum or a saddle point.

We have

$$\begin{aligned} f_x(x,y) &= 3x^2 + 6x = 3x(x+2), \\ f_y(x,y) &= 3y^2 - 36y + 81 = 3(y-3)(y-9), \end{aligned}$$

Which gives us the four critical points,

$$(0,3),\ (0,9),\ (-2,3),\ (-2,9).$$

To characterize these we use the second derivative test. We have

$$\begin{aligned} f_{xx}(x,y) &= 6x + 6, \\ f_{xy}(x,y) &= 0, \\ f_{yy}(x,y) &= 6y - 36, \end{aligned}$$

so

$$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)^2 = 36(x+1)(y-6).$$

Hence,

$$\begin{array}{lcll} D(0,3) & = & 36(1)(-3) < 0 & \Rightarrow\ (0,3) \text{ is a saddle point,} \\ D(0,9) & = & 36(1)(3) > 0 \text{ and } f_{xx}(0,9) = 6 > 0 & \Rightarrow\ (0,9) \text{ is a relative minimum,} \\ D(-2,3) & = & 36(-1)(-3) > 0 \text{ and } f_{xx}(-2,3) = -6 < 0 & \Rightarrow\ (-2,3) \text{ is a relative max,} \\ D(-2,9) & = & 36(-1)(3) < 0 & \Rightarrow\ (-2,9) \text{ is a saddle point.} \end{array}$$

5. Find the absolute maximum and minimum of the function $f(x, y)$ given in problem #B4 on the closed triangle with vertices $(-3, 4)$, $(-3, -1)$ and $(2, 4)$. (Hint: it isn't as bad as it looks. Draw a picture of the region so you can see which of the critical points of $f$ lie within the region. The derivatives on the boundaries are pretty nice; all critical points of the constrained functions have integer coördinates. On one of the sides of the triangle there is no critical point, but of course you must show that.)

   The only critical points of $f$ which lie within the triangle are $(0, 3)$ and $(-2, 3)$. Let $S_1$ denote the segment joining $(-3, 4)$ with $(-3, -1)$ (the "left" side of the triangle; let $S_2$ be the "top", joining $(-3, 4)$ with $(2, 4)$, and let $S_3$ be the remaining "diagonal", joining $(-3, -1)$ with $(2, 4)$. Define the three functions, which are the restrictions of $f$ to each corresponding segment:

$$\begin{array}{rclcll} g_1(y) & = & f(-3, y) & = & y^3 - 18y^2 + 81y + 5, & -1 \le y \le 4 \\ g_2(x) & = & f(x, 4) & = & x^3 - 18x^2 + 81x + 117, & -3 \le x \le 2 \\ g_3(x) & = & f(x, x+2) & = & 2x^3 - 9x^2 + 21x + 103, & -3 \le x \le 2. \end{array}$$

   We next find any test points that are critical points of these functions. Since

$$g_1'(y) = 3(y^2 - 12y + 27) = 3(y - 9)(y - 3),$$

   the only critical point of $g_1$ in the interval $-1 \le y \le 4$ is $y = 3$. Likewise, $g_2$ has only one critical point of interest, namely at $x = 3$. Finally,

$$g_3'(x) = 3(2x^2 - 6x + 7),$$

   which has no zeros at all, so we obtain no new critical points there. Therefore, we need only test the two original critical points of $f$, the critical points from $g_1$ and $g_2$ and the vertices of the triangle. We obtain

$$\begin{array}{lclcl} f(0, 3) & & & = & 113 \\ f(-2, 3) & & & = & 117 \\ g_1(3) & = & f(-3, 3) & = & 113 \\ g_2(3) & = & f(3, 4) & = & 159 \\ f(-3, 4) & & & = & 105 \\ f(-3, -1) & & & = & 69 \\ f(2, 4) & & & = & 125. \end{array}$$

   Therefore, the minimum is at the vertex $(-3, -1)$ and the max is at $(3, 4)$, the minimimu and maximum values being 69 and 159, respectively

6. Reverse the order of integration in the following:

$$\int_0^3 \int_1^{4-x} (x + y)\, dy\, dx$$

   (Don't evaluate the integral unless you just want to check your answer.)

   The region is the triangle with vertices $(0, 1)$, $(0, 4)$, $(3, 1)$. For each $y \in [1, 4]$, $x$ lies in $[0, 4-y]$. So our integral is

$$\int_1^4 \int_0^{4-y} (x + y)\, dx\, dy\ .$$

7. Evaluate both of the iterated integrals

$$\iint\limits_R y\,dx\,dy \quad \text{and} \quad \iint\limits_R y\,dy\,dx$$

where $R$ is the trianglular region with vertices $(-1,1)$, $(1,1)$ and $(0,2)$.

We note that the oblique "left" and "right" sides of the triangle lie along the lines $y = x+2$ and $y = 2-x$, resp. For the first integral, we see that for each $y \in [1,2]$, we have $x \in [y-2, 2-y]$, so

$$\begin{aligned}
\iint\limits_R y\,dx\,dy &= \int_1^2 \int_{y-2}^{2-y} y\,dx\,dy \\
&= \int_1^2 y[(2-y)-(y-2)]dy \\
&= \int_1^2 (4y-2y^2)dy \\
&= 4/3.
\end{aligned}$$

To go the other way, for each $x \in [-1,1]$, we have $y \in [1, x+2]$ on the left side, and $y \in [1, 2-x]$ on the right. So we split the integral into two pieces:

$$\begin{aligned}
\iint\limits_R y\,dy\,dx &= \int_{-1}^0 \int_1^{x+2} y\,dy\,dx + \int_0^1 \int_1^{2-x} y\,dy\,dx \\
&= \int_{-1}^0 \left.\frac{y^2}{2}\right|_1^{x+2} dx + \int_0^1 \left.\frac{y^2}{2}\right|_1^{2-x} dx \\
&= \int_{-1}^0 \frac{1}{2}((x+2)^2 - 1^2)\,dx + \int_0^1 \frac{1}{2}((2-x)^2 - 1^2)\,dx \\
&= \frac{2}{3} + \frac{2}{3} = \frac{4}{3}\,.
\end{aligned}$$

It might also be worth noting that we can do the above with a single integral using the absolute value function. The top half of the triangle is given by $y = 2 - |x|$ for $-1 \le x \le 1$, so the integral is

$$\int_{-1}^1 \int_1^{2-|x|} y\,dy\,dx\,.$$

Or one could note that the integral is symmetric w.r.t. the $y$-axis (due to the region *and* the integrand), so we only need

$$2\int_0^1 \int_1^{2-x} y\,dy\,dx\,.$$

8. Use polar coördinates to evaluate the integral

$$\iint\limits_R x\,dA$$

where $R$ is the region (a quarter annulus) bounded by the positive $x$-axis, the positive $y$-axis and the circles $x^2+y^2=1$ and $x^2+y^2=4$. (If you divide the result by the area of the region, you'll get the $x$-value of the centroid of the figure. By symmetry this is also the $y$-value and you'll have the centroid.)

Draw a picture. The region is that between $r=1$ and $r=2$, between $\theta=0$ and $\theta=\pi/2$. Since $x=r\ \cos\theta$, we get

$$\begin{aligned}\iint\limits_R x\,dA &= \int_1^2\int_0^{\pi/2} r\cos\theta\, r\,drd\theta \\ &= \int_1^2\int_0^{\pi/2} r^2\cos\theta\ \,drd\theta \\ &= 7/3\,.\end{aligned}$$

Incidentally, this can be used to show that the centroid is at $\left(\dfrac{28}{9\pi},\dfrac{28}{9\pi}\right)$.

## ⋆⋆⋆ Extra Credit ⋆⋆⋆

(7 points per ⋆.)

A.) (⋆) Write a *parametric representation* (with two real parameters) for the plane containing $P$, $Q$, $R$ in problem #A1.

B.) (⋆) For three arbitrary, distinct points $A, B, C \in \mathbb{R}^3$, prove that the point $(1/5)(A+2B+2C)$ lies strictly in the interior of the triangle $ABC$.

C.) (⋆) Does the point you found in problem #A2 lie within $\triangle PQR$? Justify your answer.

D.) (⋆) What kind of quadric surface does $xy=z^2$ represent? (Take an educated guess and justify it.)

E.) (⋆) Using rectangular coördinates (instead of polar coördinates), write an iterated integral for the integral in problem #B8 (in either $dx\;dy$ or $dy\;dx$ order, but what I'm looking for are the correct limits).

$$\int_0^1\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} x\;dx\;dy + \int_1^2\int_0^{\sqrt{4-x^2}} x\;dx\;dy$$

F.) (⋆$\cdots$⋆) Ask a question you wish I had asked and answer it. Make it a good one; points do vary.