Calc III Fall 2003

MATH 223 Name: ________________

Final Exam

Instructions: Answer all problems correctly. Calculators are not allowed. Each numbered problem is worth 10 points unless otherwise specified. You won't finish it. Each st⋆rred problem is extra credit and each ⋆ is worth 12 points.

1. Use Lagrange multipliers to find the maximum and minum values of the function $f(x, y, z) = x^2y^2z^2$ subject to the constraint $x^2 + 2y^2 + 3y^2 = 6$.

2. Evaluate (and simplify) the iterated integral.
$$\int_0^4 \int_{y/2}^{\sqrt{y}} (x^2 + y^2) dx\, dy$$

3. Reverse the order of integration for the previous integral. That is, find appropriate limits of integration for
$$\iint_{\mathcal{D}} (x^2 + y^2) dy\, dx$$
for the same region $\mathcal{D}$ corresponding to integral in problem #2 above. (Of course you are free to actually evaluate the new integral to check your answer, but this is not required.)

4. Use a double integral in polar coordinates to find the volume of the region that lies between the sphere $x^2 + y^2 + z^2 = 4$ and the cone $z = \sqrt{x^2 + y^2}$.

5. Use spherical coordinates to find the same volume as in problem #4.

6. Find the surface area of the spherical "cap" portion of the region in problem #4. (My advice: use the formula, in Cartesian coordinates, for the formula for surface area, then convert the integral to polar coordinates. Not that you wouldn't have done that anyway, but ...)

7. Rewrite the integral
$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z) dy\, dz\, dx$$
as an equivalent iterated intgrals of the form
$$\iiint_{\mathcal{E}} f(x, y, z) dz\, dx\, dy \quad \text{or} \quad \iiint_{\mathcal{E}} f(x, y, z) dx\, dy\, dz$$
(i.e., find appropriate limits of integration for one of these two other orders).

Old stuff starts here.

8. If a, b, c are positive, use vectors to find the area of the triangular face formed by the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ in the first octant (i.e., where x, y, z are all positive).
   [NOTE:] If you're a weenie you can let $a = 2, b = 3, c = 6$ in the above problem and subsequent problems that refer to it. But you'll miss out on the fun.

($\star$) Check your answer to problem #8 using a surface area integral. (You could also check it with Heron's formula.)

9. Find the projection, using any method you wish, of the origin onto the plane in problem #8.

10. What is the angle at the "top" vertex of the triangle in problem #8? (You've already found out the top is at $(0, 0, c)$.)

11. For the "cone function" $f(x, y) = \sqrt{x^2 + y^2}$ at the point $(3, 4, 5)$,
    (a) find the gradient of the function, and
    (b) find an equation for the tangent plane to the surface $z = f(x, y)$.

12. (*15 points*) Consider the "conical spiral" given by
    $$\mathbf{r}(t) = \langle 3t \cos t, 3t \sin t, 2t \rangle.$$
    (The following simplify quite a bit if you take a moment.)
    (a) Find the velocity vector $\mathbf{r}'(t)$ and the speed at each t.
    (b) Find the tangent vector at each t.
    (c) Find the curvature at each t.

13. Investigate the limit as $(x, y) \to (0, 0)$ of the function $f(x, y) = \dfrac{x^4 - 2x^2y^2 + y^4}{x^4 + y^4}$.

14. (*15 points*) Find the absolute maximum and minimum of the function $f(x, y) = x^2 - 2x + y^2$ on the rectangle $[-1, 2] \times [-1, 2]$. What are the minimum and maximum values of the function when restricted to the boundary of the rectangle?

$\star\star\star\star$ Extras $\star\star\star\star$

A.) ($\star$) Find the centroid of the figure in problem #4.

B.) ($\star$) Find the radius of the circle obtained by intersecting the sphere $x^2 + y^2 + z^2 = 4$ with the plane $3x + 2y + z = 1$.

C.) ($\star$) Find the angle between the surfaces in the previous problem at their intersection. (State what your interpretation of this angle is and why the problem is well-posed for this particular pair of surfaces.)

D.) ($\star \cdots \star$) Ask a question you wish I had asked and answer it. Points may vary.