Calc III — Spring 2006

MATH 223 — Name: ______________

Final Exam

Instructions: Answer all problems correctly. Each numbered problem is worth 12 points. Each st⋆rred problem is "extra credit" and each ⋆ is worth 6 points. Do as many problems as you wish, but no grade higher than 115 points (including a curve, if any) will be awarded on the test. Calculators are not allowed. *Clarity counts, so be clear. Points will be lost if I have to guess what you're doing. This is especially true when you are asked to prove something.*

Part A. These are items from the Early Tritestocene Era, more or less. Do as many as you wish, but I'll only count up to 60 points from this batch.

1. (4 points) Consider the three points $P = (-1, 1, 2)$, $Q = (1, 2, 3)$ and $R = (1, 0, 4)$. (These points will be refrerred to again in subsequent problems.) What point lies 3/4 the way from the point P to the point R? $\boxed{4}$

(⋆) What points lie one unit from P on the line joining P and R?

2. (4 points) Find the area of triangle PQR. $\boxed{8}$

3. (4 points) Find a parameterization for the *ray* (not the line, not a line segment) starting at P and passing through R. $\boxed{12}$

4. (4 points) Give an equation in standard form for the plane contining P,Q and R. $\boxed{16}$

5. (4 points) Find *parametric equations* for the plane containing P,Q,R. (There are *two* parameters, you'll recall.) $\boxed{20}$

(⋆) Find a parameterization for the triangle whose vertices are P,Q,R. (This amounts to using the result of the previous problem and finding a suitable restriction on the two parameters.)

6. (4 points) Write an equation for the sphere for which P and Q are the endpoints of a diameter. $\boxed{24}$

7. (8 points) Find the *reflection* of the point Q with respect to the line containing P and R. $\boxed{32}$

8. (4 points) Identify by name (e.g., "ellipsoid", "Gabriel's horn", "Dr. Mabry's chair", etc.) the surfaces defined by each of the following equations. $\boxed{36}$

   (a) $x^2 + 2z^2 = y^2$
   (b) $x^2 + 2z^2 = y^2 + 1$
   (c) $x^2 + 2z^2 = y + 1$
   (d) $x^2 + 2z^2 = 1$

9. (4 points) Prove that for any twice-differentiable space curve $\mathbf{u}(t)$, $\boxed{40}$

$$\frac{d}{dt}\left(\mathbf{u}'(t) \times \mathbf{u}(t)\right) = \mathbf{u}''(t) \times \mathbf{u}(t).$$

10. (6 points) Consider the planes $x + 2y + z = 6$ and $2x - y - z = 0$. $\boxed{46}$

   (a) Find the angle between these planes.

   (b) Find a parameterization for the line lying at the intersection of the planes.

11. (6 points) The surfaces $x^2 + 2y^2 = 9$ and $z = (x-1)^2 + y^2$ intersect to form a curve. The point $(1, 2, 4)$ lies on this curve. Give a parametric representation of the slope of the line tangent to the curve at the point. $\boxed{52}$

12. (6 points) Consider the curve $\mathbf{r}(t) = \langle 2\cos t, 2\sin t + t, 2t + 1\rangle$. $\boxed{58}$

   (a) Calculate the curvature $\kappa(t)$.

   (b) Calculate the tangential and normal components, $a_T(t)$ and $a_N(t)$, of the acceleration vector.

13. (9 points) For each of the following, decide whether or not $f(x, y)$ has a limit at $(0, 0)$ and *prove your answer*. (Do verbalize coherently when proving things — displaying a jumble of equations and expecting me to assemble it won't get you full credit. *Speak* to me.) $\boxed{67}$

   (a) $f(x, y) = \dfrac{x - 2y}{3x^2 + 2y^2}$

   (b) $f(x, y) = \dfrac{x^2 - 2y^2}{3x^2 + 2y^2}$

   (c) $f(x, y) = \dfrac{x^3 - 2y^3}{3x^2 + 2y^2}$

14. (6 points) The point $(-2, 1, 0)$ lies on the surface defined by $(x^2 - 3y)e^{2z} = 1$ (but don't take *my* word for it). $\boxed{73}$

   (a) Find $\dfrac{\partial z}{\partial x}$ at that point.

   (b) Find an equation for the tangent plane at that point.

Part B. The Late Stressaceous Era. Go for it, I'll take as many as you can dish.

15. (8 points) Suppose $f(x, y)$ is a differentiable function and that its partials at the point $(1, 2)$ are given by $f_x(1, 2) = 3$ and $f_y(1, 2) = -1$. Calculate $h'(-1)$ if $h(t) = f(t^3 + 2, t^2 + 1)$. $\boxed{81}$

16. (8 points) For the function $f(x, y)$ in the previous problem, calculate the directional derivative $D_{\mathbf{u}}f(1, 2)$ when $\mathbf{u} = \langle \cos\frac{\pi}{6}, \sin\frac{\pi}{6}\rangle$. $\boxed{89}$

17. (8 points) Find all saddle points, local maxima and local minima (if there are any of these) for the function $f(x, y) = 8x^2 + y^2 - 4x^2y$. $\boxed{97}$

18. (10 points) Find the absolute max and min of the function $f(x, y) = xy + y^2 - y$ on the closed triangles whose vertices are $(2, 0), (0, 2), (0, -2)$. $\boxed{107}$

19. (8 points) Evaluate the integral $\displaystyle\int_0^1 \int_{x^2}^{2-x} (x^2 - 2y)\,dy\,dx$. $\boxed{115}$

20. (8 points) Write the result of reversing the order of integration in the previous problem. You needn't evaluate the integral or integrals obtained (unless you want to check your answer). $\boxed{123}$

21. (8 points) Use a double integral in polar coordinates to find the area inside the circle $r = 4\sin\theta$ and outside the circle $r = 2$. $\boxed{131}$

22. (4 points) Write an integral that gives the area of the portion of the surface $z = x^2y^4$ that lies above the triangle with vertices $(2, 0), (0, 2), (0, -2)$. $\boxed{135}$

23. (8 points) Evaluate the integral $\int_0^1 \int_0^z \int_0^y xy^2z^3\, dx\, dy\, dz$. $\boxed{143}$

⋆ ⋆ ⋆ ⋆ Extras ⋆ ⋆ ⋆ ⋆

A. (⋆) Prove that the area of a simple plane quadrilateral $ABCD$ is given by $(1/2)||\mathbf{AC} \times \mathbf{BD}||$.

B. (⋆⋆) Find the minimum and maximum curvature of the curve given in problem #12.

C. (⋆) Use the definition of differentiability, prove that the function $f(x, y) = x^2y^3$ is differentiable at each point. (This is fairly computation-intensive unless you use some intelligent algebraic tricks.)

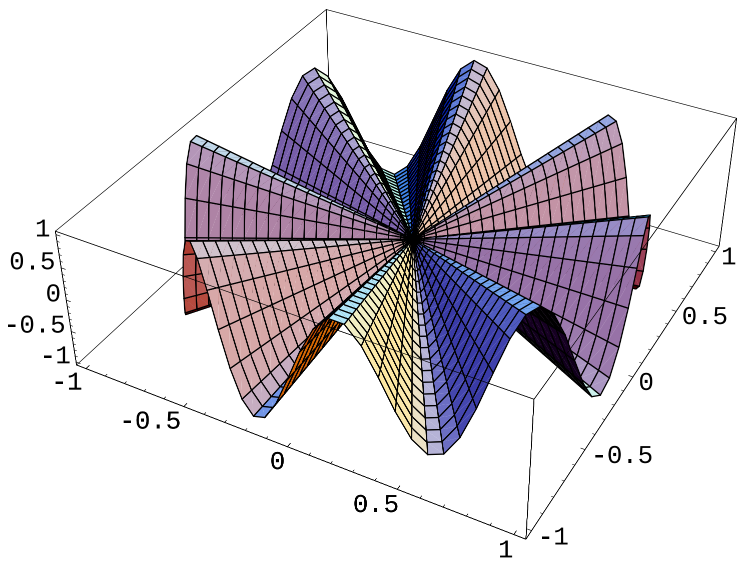
D. (⋆) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be fixed vectors with $\mathbf{a} \neq \mathbf{0}$. Show that the curve $\mathbf{r}(t) = \mathbf{a}t^2 + \mathbf{b}t + \mathbf{c}$ lies entirely in a plane. Find an equation for the plane.

E. (⋆) State clearly and completely **Clairaut's Theorem** concerning mixed partials.

F. (⋆) Find $f_y(0, 0)$ if $f(x, y) = \sqrt[7]{x^7 + y^7}$. Show all steps, which count.

G. (⋆) Give an example of a function $f(x, y)$ whose mixed partial derivatives f_{xy} and f_{yx} exist at a point but are not equal there.

H. (⋆) Give an equation for an "octopus's saddle". Personally, I'd use cylindrical coordinates. (Unfair as it might sound, for credit you'll need to satisfy *my* idea of such a saddle, as opposed to arguing you can plop an octopus on a paraboloid or some such thing. The octopus has eight arms and will need a separate lodging for each, in order to be comfy.)



I. ($\star$) Find the distance between the two lines parameterized below.

$$\mathbf{r}_1(t) = \langle 2 - t, 3 + 2t, t + 1 \rangle$$

$$\mathbf{r}_2(s) = \langle 2s + 1, s - 1, 3 \rangle$$

J. ($\star$) Find the centroid of the region of space lying between the surface and region described in problem #22.

K. ($\star$) Without actually evaluating the integrals involved, find the centroid of the patch of *surface* described in problem #22. (You'll assume the surface has constant planar density.)

L. ($\star$) Reverse the order of integration in problem #23. This means you need to find the limits of integration for $\displaystyle\iiint_D xy^2z^3 \; dz \; dy \; dx$, in that order. The region is a tetrahedron, so it isn't all that bad. Your answer should match your earlier one, of course.

M. ($\star \cdots \star$) Ask a question you wish I had asked and answer it. (It may have multiple parts.) Points vary greatly depending on the difficulty of the question (and the correctness of the answer). Very few points (if any) will be awarded for a problem that is essentially represented elsewhere on the test. (In other words, no double-dipping.)