Calc III — Spring 2013

MATH 223 — Name: ______________

Final Exam

Instructions: Answer all problems correctly. Write all responses on separate paper (provided) and *keep the problems in their numbered order.*

No calculators or other electronic gadgets may be used. However, you needn't spend much time simplifying answers unless instructed to do so. *Unless instructed to do so, just "set up" definite integrals, leaving them unevaluated.*

Attempt as many problems as you wish, even though no grade higher than 115 points (including a curve, if any) will be awarded on the exam. (But that's a lot!) Furthermore, the exam is divided into sections, each with separate maximum number of allowed points.

Each st$\star$rred problem is "extra credit" and each $\star$ is worth 8 points. These are not counted toward maximum allowable points—do as many as you wish.

A. The "T1" section. The numbered problems in this section (i.e., the problems marked by a number, as opposed to a letter or $\star$) are worth 6 points each. At those values the maximum number of points that can be earned in this section is 30; any points over that will be counted at 50%. (For example, 40 points becomes 35.) This 50% reduction applies in all sections except the "T4" and extra-credit sections, where you can go wild.

In all problems on this exam, O will refer to the origin, and P, Q and R will always refer to these same points:

$$P = (0, 0, 1),\ Q = (2, 2, 0) \text{ and } R = (-3, 2, 1).$$

1. What point lies 3/5 the way from the point R to the point Q? Simplify.
2. Find the point S that makes $PSQR$ (yes, in that order) a parallelogram. Simplify.
3. Find the area of the triangle PQR. Find the volume of tetrahedron $OPQR$. Simplify.
4. Write simplified equations in standard form for
   (a) the plane π_{PQR} containing triangle PQR, and
   (b) the plane parallel to plane π_{PQR}, but passing through the point $(0, 0, 10)$.
5. Equations of lines and segments...
   (a) Find parametric equations and symmetric equations for the line passing through the points P and R.
   (b) Write a parameterization for the *line segment* $\overline{PQ}$ joining points P and Q.
6. Find the distance from the point O to the plane π_{PQR}.

($\star$) While you're at it, find (a) the projection of the *point* O onto plane π_{PQR}. and (b) the reflection of the point O with respect to plane π_{PQR}.

7. Let T denote the point $(0, 0, 10)$. Do O and T lie on opposite sides of plane π_{PQR}? Prove your result.

B. The "T2" section. The numbered problems in this section are worth 6 points each. At those values the maximum number of points that can be earned in this section is 30.

1. Identify by name the surfaces described (e.g., ellipsoid, cone, paraboloid, etc.).
   (a) $x^2 - 3z^2 = 2y^2 + 1$
   (b) $x^2 - 3z^2 = 2y^2$
   (c) $x^2 - 3z^2 = 2y - 1$
   (d) $x^2 - 3z + 2y^2 = 1$
2. A particle's position is given by $\mathbf{r}(t) = \langle 2t^2 - 1, 4t, 3\rangle$. How far does the particle travel *along its arc* between $t = 1$ and $t = 3$?
3. Find a parameterization for the curve formed by the intersection of the two surfaces

   $$z = 5\sqrt{x^2 + y^2} \quad \text{and} \quad 2x + 3y - z = -4.$$

   Hint: Write x and y in polar coordinates r and θ in the usual way. Set the z-values equal and solve for r in term of θ. Now you're almost finished. (But why am I telling you all this? Answer: In case you've never seen a conic section given in terms of polar coordinates. In this case the curve is an ellipse.)
4. The function $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$ traces a space curve. Find a parameterization for *the line tangent to the curve* at the point $\mathbf{r}(1)$.
5. Find the curvature at $t = 1$ of the curve given by $\mathbf{r}(t)$ in problem #B4.
6. Two particles travel along the following curves, parameterized by time t.

   $$\mathbf{r}_1(t) = \langle t^2, 2 + t, t^2 + 2\rangle \quad \text{and} \quad \mathbf{r}_2(t) = \langle t(t-1), 2t, 3t - 1\rangle$$

   Find

   i. all collisions and
   ii. all intersections

   of these two curves (if any). Give the (x, y, z) coordinates of the solutions. Show all relevant steps that must be used to solve problems of this type, even if you can use shortcuts or find the answers by inspection in this particular problem. You're trying to convince me that you can deal with more complicated examples.

C. The "T3" section. The numbered problems in this section are worth 7 points each. At those values the maximum number of points that can be earned in this section is 35.

1. Be precise:
   (a) Let $f(x,y)$ be defined on a region $D \subset \mathbb{R}^2$. Write the definition of the statement
   $$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$
   (b) State *Clairaut's Theorem.*
2. Find $f_{xy}(x,y,z)$ when $f(x,y,z) = x\sin(x^2y^3z^4)$.
3. Use the definition of differentiability to prove that the function $f(x,y) = x^2 - 2xy$ is differentiable at (a,b).
4. For each of the following, state whether or not $f(x,y)$ has a limit at $(0,0)$ and calculate the limit if it does exist. Justify your results.
   (a) $f(x,y) = \dfrac{x(x^2+xy+y^2)}{x^2+2y^2}$
   (b) $f(x,y) = \dfrac{x^2+xy+y^2}{x^2+2y^2}$
5. Suppose $f(x,y)$ is differentiable at $(0,0)$, with $f_x(0,0) = 2$ and $f_y(0,0) = 3$. Suppose further that $h(t)$ is a differentiable function such that $h(0) = 0$ and $h'(0) = 7$. Compute $g'(0)$, where
   $$g(u) = f(2h(u), 3(h(u))^2).$$
6. True or False? If $f : \mathbb{R}^2 \to \mathbb{R}$ and both of $f_x(0,0)$ and $f_y(0,0)$ exist, then f is continuous at $(0,0)$. Justify your answer.
7. Let $f(x,y) = 3xy^2$. Find the directional derivative of f at $(0,0)$ in the direction of the point $(1,1)$.

D. The "T4" section. No limit on the points that can be earned in this section; the problems here are worth 8 points each.

1. Let $f(x,y) = 3x^3 - xy^2 + y^2$. Write a simplified equation, in standard form, for the tangent plane to the surface $z = f(x,y)$ when $(x,y) = (0,1)$.
2. Again, let $f(x,y) = 3x^3 - xy^2 + y^2$.
   a.) Find all critical points of f.
   b.) Use the second derivative test (if possible) to classify each critical point as a relative maximum, a relative minimum, or a saddle point.
3. (*Counts as two problems.*) Again, let $f(x,y) = 3x^3 - xy^2 + y^2$. Find the absolute maximum and minimum of f in (and on) the triangle having vertices $(0,0)$, $(0,2)$, and $(2,2)$.
4. Evaluate (and simplify) the integral $\displaystyle\int_0^1 \int_{\sqrt{y}}^{2-y} x\,dx\,dy.$
5. Write the result of reversing the order of integration in the previous problem. You needn't evaluate the integral(s) obtained (unless you want to check your answer). I strongly recommend a good picture for guidance.

$\star \quad \star \quad$ Extras $\quad \star \quad \star$

($\star$1) Write a parameterization of the line in plane π_{PQR} that is the perpendicular bisector of the line segment joining points P and Q.

($\star$2) In plane π_{PQR}, write a parameterization of the circle centered at point P and passing through point Q.

($\star$3) For each of the following, state whether or not $f(x,y)$ has a limit at $(0,0)$ and calculate the limit if it does exist. Justify your results. (Warning: One of the limits exists, the other does not.)

(a) $f(x,y) = \dfrac{x^3+y^3}{x^2+3xy+y^2}$

(b) $f(x,y) = \dfrac{x^3+y^3}{3x^2+xy+3y^2}$

($\star$4) You are stuck in the plane π_{PQR}, presently where $x=0$ and $y=0$. Give a point on the plane toward which you should move in order that your distance from the xy-plane changes most rapidly (assuming that your speed is constant).

($\star$5) Write an integral that gives the *surface area* of the portion of the paraboloid $z = x^2+y^2$ that lies above the unit square $[0,1]\times[0,1]$.

($\star$6) Use a double integral in polar coordinates to find the area of the intersection of the circles $r = 2\sin\theta$ and $r = 2\cos\theta$.

($\star$7) (This is to get you to calculate a Jacobian, nothing more.) Suppose one has an integral over a region $\mathcal{D}$ in the plane:

$$\iint_{\mathcal{D}} f(x,y)dA,$$

where dA is given in rectangular coordinates, i.e., as $dx\,dy$ or $dy\,dx$. It is suggested that new variables u, v be used, where $x = 2u - v^2$ and $y = uv - 2v$. Use the Jacobian to compute dA in terms of u, v, du, dv.

($\star$8) Ask a question you wish I had asked and answer it. Points vary depending on the difficulty of the question (and the correctness of the solution). Very few points (if any) will be awarded for a problem that is essentially represented elsewhere on the test. (In other words, no repeats.)