Calc III
MATH 223

Fall 2003
Name: ____________

Test #1

Instructions: Answer all problems correctly. Calculators are not allowed. Each numbered problem is worth 7 points. Each st$\star$rred problem is extra credit, and each $\star$ is worth 5 points.

1. Consider the three points $P = (-1, 1, 0)$, $Q = (1, 0, 2)$ and $R = (0, 1, 1)$. Find the distance between points P and Q. (In the subsequent problems, P, Q and R will always refer to these same points.)

$$||\mathbf{PQ}|| = ||Q - P|| = ||\langle 2, -1, 2\rangle|| = \sqrt{2^2 + (-1)^2 + 2^2} = 3.$$

2. What is the midpoint of the segment $\overline{PQ}$? What point lies 3/4 the way from the point P to the point Q?

The midpoint is the "average point", $(P + Q)/2 = \left\langle \frac{-1+1}{2}, \frac{1+0}{2}, \frac{0+2}{2} \right\rangle = \left\langle 0, \frac{1}{2}, 1 \right\rangle$. The easiest way to find the other point is as

$$P + (3/4)\mathbf{PQ} = P + (3/4)(Q - P) = (3/4)Q + (1/4)P = \cdots = \left\langle \frac{1}{2}, \frac{1}{4}, \frac{3}{2} \right\rangle.$$

3. Find the coordinates of the point obtained by rotating Q about the z-axis counterclockwise through $30°$.

Leave the z-coordinate alone and just rotate $(x, y) = (1, 0)$ by $30°$ about the origin. The formula for the new point (x', y') is given by

$$x' = x \cos 30° - y \sin 30°, \qquad y' = x \sin 30° + y \cos 30°,$$

which works out to simply $(x', y') = (\cos 30°, \sin 30°) = (\sqrt{3}/2, 1/2)$. (I didn't notice it turned out so easy. The cool formula isn't really needed.) So the rotated point we want is

$$Q' = (\sqrt{3}/2,\ 1/2,\ 2).$$

4. A sphere is centered at point P and passes through Q. Write an equation for the sphere.

The equation can be written $||(x, y, z) - P|| = ||Q - P||$ but it looks nicer when you square both sides to get

$$(x + 1)^2 + (y - 1)^2 + z^2 = 9.$$

5. What is the angle at vertex Q in the triangle formed from the points P, Q and R? (Obviously, you'll have to leave your answer in "calculator-ready form".)

   Since $\mathbf{QP} \cdot \mathbf{QR} = ||\mathbf{QP}||||\mathbf{QR}|| \cos\theta$, we have

$$\cos\theta = \frac{\langle -2, 1, -2\rangle \cdot \langle -1, 1, -1\rangle}{||\langle -2, 1, -2\rangle||\ ||\langle -1, 1, -1\rangle||} = \frac{5}{(3)(\sqrt{3})},$$

   so $\theta = \arccos\left(\frac{5}{3\sqrt{3}}\right)$.

6. Find a vector that bisects the angle at vertex Q of the triangle PQR.

   One bisector is $||\mathbf{QP}||\mathbf{QR} + ||\mathbf{QR}||\mathbf{QP}$, which works out to

$$\begin{aligned} ||\langle -2,\ 1,\ -2\rangle||\,\langle -1,\ 1,\ -1\rangle \ &+\ ||\langle -1,\ 1,\ -1\rangle||\,\langle -2,\ 1,\ -2\rangle \\ &=\ 3\,\langle -1, 1, -1\rangle + \sqrt{3}\,\langle -2, 1, -2\rangle \\ &=\ \left\langle -3 - 2\sqrt{3},\ 3 + \sqrt{3},\ -3 - 2\sqrt{3}\right\rangle. \end{aligned}$$

7. Find the area of the triangle PQR.

   It's just $\frac{1}{2}||\mathbf{QP} \times \mathbf{QR}||$. Since

$$\mathbf{QP} \times \mathbf{QR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & -2 \\ -1 & 1 & -1 \end{vmatrix} = \langle 1, 0, -1\rangle\,,$$

   the area is $\frac{1}{2}\sqrt{2}$.

8. Write an equation in standard form for the plane containing triangle PQR. Write an equation for the plane parallel to this plane but passing through the origin.

   The first plane has a normal vector given by $\mathbf{n} = \mathbf{QP} \times \mathbf{QR} = \langle 1, 0, -1\rangle$, so an equation for the plane is given by

$$\mathbf{n} \cdot ((x, y, z) - Q) = 0,$$

   or

$$\langle 1, 0, -1\rangle \cdot \langle x - 1, y, z - 2\rangle = 0,$$

   or

$$x - z = -1.$$

   Any plane parallel to this form will have equation $x - z = d$, so the one passing through the origin must be

$$x - z = 0.$$

9. Find parametric equations and the symmetric equations for the line passing through the points P and Q.

   Parametric:

$$\{P + t\,\mathbf{PQ} : t \in \mathbb{R}\} = \{(-1 + 2t, 1 - t, 2t) : t \in \mathbb{R}\},$$

   or

$$(x, y, z) = (2t - 1,\ -t + 1,\ 2t), \quad t \in \mathbb{R}.$$

   Symmetric:

$$\frac{x + 1}{2} = 1 - y = \frac{z}{2}.$$

10. Let S denote the point $(1, 2, 3)$ and let $\mathbf{v}$ be the vector $\langle -1, 3, 0\rangle$. Find the projection of the vector $\mathbf{SP}$ onto $\mathbf{v}$. (We'll refer to these S, and $\mathbf{v}$ again.)

    The projection is

$$\text{proj}_{\mathbf{v}}\mathbf{SP} = \frac{\mathbf{v} \cdot \mathbf{SP}}{||\mathbf{v}||^2}\,\mathbf{v} = \cdots = \left\langle \frac{1}{10}, -\frac{3}{10}, 0 \right\rangle.$$

11. Let L denote the line passing through the point S and parallel to the vector $\mathbf{v}$. Find the distance from L to the point P. (We'll refer to this L again.)

    The distance is the length of the orthogonal projection of $\mathbf{SP}$ onto $\mathbf{v}$. The orthogonal projection is given by

$$\mathbf{SP} - \text{proj}_{\mathbf{v}}\mathbf{SP} = \cdots \left\langle -\frac{21}{10}, -\frac{7}{10}, -3 \right\rangle,$$

    which has length $\sqrt{\frac{139}{10}}$.

12. Write an equation for the plane containing the point P and perpendicular to the line L.

    The line L is parallel to $\mathbf{v}$, which we can take as a normal vector to the plane. An equation of the plane is therefore given by

$$\mathbf{v} \cdot ((x, y, z) - P) = 0,$$

    or

$$\langle -1, 3, 0\rangle \cdot \langle x + 1, y - 1, z\rangle = 0,$$

    which gives

$$x - 3y = -4.$$

13. Write an equation for the plane containing the point P and *also containing* the line L.

    A normal vector to the plane will be any vector $\mathbf{n}$ perpendicular to both $\mathbf{SP}$ and $\mathbf{v}$, so we can use the cross-product to get

$$\mathbf{n} = \mathbf{SP} \times \mathbf{v} = \cdots = \langle 9, 3, -7\rangle.$$

    We can now write the plane as

$$9x + 3y - 7z = 9(-1) + 3(1) - 7(0) = -6,$$

    where we used the point P to determine the righthand side.

14. Find *parametric equations* for the plane containing P,Q,R.

From class:

$$\pi = \{P + s\,\mathbf{PQ} + t\,\mathbf{PR} : s,t \in \mathbb{R}\}.$$

So here a parameterization is

$$(x,y,z) = (2s+t-1, 1-s, 2s+t), \quad s,t \in \mathbb{R}.$$

15. At what point does the line L intersect the plane containing P,Q,R?

A parameterization for L is

$$(x,y,z) = S + t\,\mathbf{v}, \quad t \in \mathbb{R},$$

or

$$(x,y,z) = (-t+1, 3t+2, 3), \quad t \in \mathbb{R}.$$

We already have an equation for the plane,

$$x - z = -1,$$

so just substitute to get

$$(-t+1) - (3) = -1$$

and solve to get $t = -1$, which corresponds to the point $(2,-1,3)$.

16. Write a parameterization for the *ray* $\vec{PQ}$.

In general, it is just $\{P + t\,\mathbf{PQ} : t \geq 0\}$ (where the $t \geq 0$ part is the essential feature of the ray, as opposed to just the line). We already calculated this for the line, so here we get

$$\{(-1+2t, 1-t, 2t) : t \geq 0\}.$$

17. Prove that the midpoints of the sides an arbitrary quadrilateral $ABCD$ always form a parallelogram.

A quadrilateral $ABCD$ is a parallelogram iff $\mathbf{AB} = \mathbf{DC}$. If we let $A'B'C'D'$ denote the quadrilateral of midpoints of $ABCD$, where

$$A' = ((A+B)/2, \quad B' = (B+C)/2, \quad C' = (C+D)/2, \quad D' = (D+A)/2,$$

then it suffices to show that $\mathbf{A'B'} = \mathbf{D'C'}$. This is fairly immediatem, since

$$\begin{aligned}
\mathbf{A'B'} - \mathbf{D'C'} &= (B' - A') - (C' - D') \\
&= \frac{1}{2}[((B+C) - (A+B)) - ((C+D) - (D+A))] \\
&= \mathbf{0}
\end{aligned}$$

18. Prove that the area of a simple plane quadrilateral $ABCD$ is given by $(1/2)||\mathbf{AC} \times \mathbf{BD}||$.

   The easiest trick I know is to chop the quad into two triangles and compute each area. The cross-=product can then be used for each triangle *taking care to obtain the same orientation for the vectors* so that the cross-products involved point in the same direction. I'll show you in class.

$\star \quad \star \quad \star \quad \star \quad \star$

A. ($\star$) Are any of the angles in triangle PQR obtuse? Justify your answer.

   An angle of a triangle is obtuse iff it is greater than 180°, which is true iff the cosine of the angle is negative, which is true iff the dot product used to compute it is negative. You can check that the angle at R is obtuse.

B. ($\star$) Find a parametric representation for the line along which the planes $x + 2y + 3z = 6$ and $4x - 2y + z = 4$ intersect.

   Eliminate any variable from the system of TWO equations in the three unknowns. I'll get rid of y by adding the equations:

$$5x + 4z = 10.$$

   Let x be the free parameter. In fact, I'll let $x = t$ to make it look standard. We get $z = (10-5t)/4$. Plugging these into the equation for the first plane and solving for y gives $y = (11t + 6)/8$. We can now write

$$(x, y, z) = \left(t, \frac{11}{8}t + \frac{3}{4}, -\frac{5}{4}t + \frac{5}{2}\right).$$

   Note that the solution is not unique. For example, one could use y or z as the free parameter. But all of the solutions will be unique up to a linear change of parameter. That is, any two parameterizations can be transformed into one another with a linear change of variable. A nicer version can be obtained, for instance, by letting $t = 8s$, to get

$$(x, y, z) = \left(8s, 11s + \frac{3}{4}, -10s + \frac{5}{2}\right).$$

C. ($\star$) Find the volume of the tetrahedron formed by intersecting the first plane of problem #8 with the three coordinate planes (the xy, xz and yz planes).

   RATS! There isn't one because one of the coordinates of normal to the plane of PQR is zero. This means that the plane fails to intersect one of the coordinate axes and no such tetrahedron is formed. I should have checked this. However, as we mentioned in class, in the event that a plane $ax + by + cz = d$ has none of a, b, c equal to zero, then the volume is $\left|\dfrac{d^3}{6abc}\right|$

D. ($\star$) Find a formula for the projection of a vector $\mathbf{w}$ onto a plane $ax + by + cx + d = 0$. (You'll have to intuit on your own what such a projection must mean.)

E. ($\star \cdots \star$) Ask a question you wish I had asked and answer it. Points may vary.