Calc III — Fall 2003

MATH 223 — Name: ________________

Test #2

Instructions: Answer all problems correctly. Calculators are not allowed. Each st⋆rred problem is extra credit and each ⋆ is worth 5 points. Leave any integrals in unevaluated form. The acceleration due to gravity at the surface of the earth is approximately 32 ft/sec/sec. Planck's constant is about $6.62606810^{-34}\text{m}^2\text{kg/s}$ and one electron volt is approximately 1.6021764610^{-19} Joules. I'll award a maximum of 120 points on this exam (including a curve, if any.)

1. (12 points) Give the name (e.g., elliptic paraboloid, hyperboloid of two sheets, etc.) of each of the following surfaces in $\mathbb{R}^3$.

   (a) $2x^2 + 3z^2 = 5$

   elliptical cylinder (y runs free)

   (b) $2x^2 + 3z^2 = 5(y-1)$

   elliptical paraboloid (elliptical cross sections for each $y = k$, $k > 1$)

   (c) $2x^2 + 3z^2 = 5y^2 - 1$

   two-sheeter (elliptical cross sections for each $y = k$, where $k < -1\sqrt{5}$ or $k > 1/\sqrt{5}$)

   (d) $2x^2 + 3z^2 = 5(y-1)^2$

   elliptical cone (elliptical cross sections for each $y \neq 1$)

   (⋆) $(2x + 3z)^2 = 5y^2$

   pair-o-planes (rewrite as $2x + 3z = \pm\sqrt{5}y$)

2. (12 points) Let $\mathbf{r}(t) = \langle 2\cos\pi t, -2\sin\pi t, 4t+2\rangle$ denote a space curve in $\mathbb{R}^3$. Find the tangent vector, $\mathbf{T}(t)$, to the curve.

   $\mathbf{r}'(t) = \langle -2\pi\sin\pi t, -2\pi\cos\pi t, 4\rangle$, so $\|\mathbf{r}'(t)\| = \sqrt{4\pi^2 + 16} =: C$ Hence

$$\mathbf{T}(t) = \mathbf{r}'(t)/\left\|\mathbf{r}'(t)\right\| = \frac{\langle -2\pi\sin\pi t, -2\pi\cos\pi t, 4\rangle}{C}.$$

3. (12 points) Let $\mathbf{r}(t)$ be as in problem #2. Find the normal vector, $\mathbf{N}(t)$.

   $\mathbf{T}'(t) = \frac{1}{C}\langle -2\pi^2 \cos \pi t, 2\pi^2 \sin \pi t, 0\rangle$, so $\|\mathbf{T}'(t)\| = 2\pi^2/C$. Hence

   $$\mathbf{N}(t) = \mathbf{T}'(t)/\|\mathbf{T}'(t)\| = \langle -\cos \pi t, \sin \pi t, 0\rangle.$$

   Note! I think I neglected a sign change negative sign on the final expression when we did this in class.

4. (10 points) Find the speed, ds/dt, of this same $\mathbf{r}(t)$.

   We already have it: $\frac{ds}{dt} = \|\mathbf{r}'(t)\| = \sqrt{4\pi^2 + 16} = C$

5. (8 points) Find the total distance, *along the curve* in problem #2, between the points $(2, 0, 2)$ and $(0, 2, -8)$.

   Use the z-component to solve for t, since the x and y components are not one-to-one. For the first point we have $4t + 2 = 2$, so $t = 0$; for the second, $4t + 2 = -8$, so $t = -5/2$. Thus

   $$s(t) = \left| \int_0^{-5/2} \frac{ds}{dt}\, dt \right| = \left| \int_0^{-5/2} C\, dt \right| = 5C/2.$$

6. (6 points) Let $\mathbf{r}(t)$ be as in problem #2. Find the tangential and normal components, $a_T(t)$ and $a_N(t)$, of the acceleration vector.

   We have $\mathbf{a} = \mathbf{r}''(t) = a_T\mathbf{T} + a_N\mathbf{N} = \frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)\kappa\ \mathbf{N}$, or $a_T = \frac{d^2s}{dt^2} = 0$ (since $\frac{ds}{dt} = C$), which means $\mathbf{r}''(t) = a_N\mathbf{N}$. But

   $$\mathbf{r}''(t) = \langle -2\pi^2 \cos \pi t, 2\pi^2 \sin \pi t, 0\rangle,$$

   $a_N = 2\pi^2$.

7. (10 points) Let $\mathbf{p}(t) = \langle t^2, t, 4t - 2t^2\rangle$ denote a space curve in $\mathbb{R}^3$. Find the curvature, $\kappa(t)$, at each point of the curve. Where is the curvature maximum?

   $\mathbf{p}'(t) = \langle 2t, 1, 4 - 4t\rangle$, $\mathbf{p}''(t) = \langle 2, 0, -4\rangle$, so

   $$\mathbf{p}'(t) \times \mathbf{p}''(t) = \langle -4, 8, -2\rangle.$$

   This gives us

   $$\begin{aligned} \kappa(t) &= \frac{\|\mathbf{p}'(t) \times \mathbf{p}''(t)\|}{\|\mathbf{p}'(t)\|^3} \\ &= \frac{\|\langle -4, 8, -2\rangle\|}{\|\langle 2t, 1, 4 - 4t\rangle\|^3} \\ &= \frac{2\sqrt{21}}{\sqrt{\left((2t)^2 + 1^2 + (4 - 4t)^2\right)^3}} \\ &= \frac{2\sqrt{21}}{\sqrt{(17 - 32t + 20t^2)^3}}. \end{aligned}$$

I think I made a slight error in class. Now this is maximized when $17 - 32t + 20t^2$ is minimized, which occurs when $t = 32/40 = 4/5$, The point in question is $\mathbf{p}(4/5) =$ (whatever).

8. (8 points) $\mathbf{p}(t)$ be as in the previous problem. At what point does the line tangent to the curve, at $t = 1$, intersect the xy-plane?

The intersection would be where the line given by $\mathbf{p}(1) + t\mathbf{p}'(1)$ intersects the xy-plane, i.e., when $z = 0$. But here, from the previous problem, $\mathbf{p}'(t) = \langle 2t, 1, 4 - 4t\rangle$, so $\mathbf{p}'(1) = \langle 2, 1, 0\rangle$, which has zero z-component. So this tangent line is parallel to the xy-plane! (Curses!) And the z-component of $\mathbf{p}(1)$ is not zero, so there is *no* intersection. Summing it up, the tangent line is given by

$$\mathbf{p}(1) + t\mathbf{p}'(1) = \langle 1, 1, 2\rangle + t\langle 2, 1, 0\rangle = \langle 2t + 1, t + 1, 2\rangle,$$

for which $z(t) \equiv 2$ is never zero.

9. (10 points) Suppose that a particle moves along a twice differentiable space curve with constant speed. Prove that its velocity and acceleration vectors are perpendicular.

The old dot-product trick... Let $\mathbf{r}(t)$ be the curve. Since

$$ds/dt = \|\mathbf{r}'(t)\| = \text{constant},$$

we have $\|\mathbf{r}'(t)\|^2 = \text{constant}$, so

$$0 = \frac{d}{dt}\|\mathbf{r}'(t)\|^2 = \frac{d}{dt}(\mathbf{r}'(t) \cdot \mathbf{r}'(t)) = 2\mathbf{r}'(t) \cdot \mathbf{r}''(t).$$

This shows $\mathbf{r}'(t) \perp \mathbf{r}''(t)$.

10. (5 points) If $\mathbf{r}(t)$ is any twice differentiable space curve, show that

$$\frac{d}{dt}(\mathbf{r}(t) \times \mathbf{v}(t)) = \mathbf{r}(t) \times \mathbf{a}(t),$$

where, as usual, $\mathbf{v}(t) = \mathbf{r}'(t)$ and $\mathbf{a}(t) = \mathbf{r}''(t)$.

Just apply the product rule for cross products to get

$$\frac{d}{dt}(\mathbf{r}(t) \times \mathbf{v}(t)) = \mathbf{r}'(t) \times \mathbf{v}(t) + \mathbf{r}(t) \times \mathbf{v}'(t) = \mathbf{v}(t) \times \mathbf{v}(t) + \mathbf{r}(t) \times \mathbf{a}(t) = \mathbf{0} + \mathbf{r}(t) \times \mathbf{a}(t).$$

11. (10 points) A projectile is launched from ground level (on earth) and lands, again at ground level, 2000 feet away. At what angle was the projectile launched and at what initial speed?

Aaarrrggghhhh. I meant to give you a maximum height. Obviously, many parabolas can be drawn that have a base of 2000 feet. Oh well. Tell you what: be able write up, in five minutes of class time, the solution to the following minor variation and I'll count it as (up to) 10 more points on T2.

A projectile is launched from ground level on a planet with a constant acceleration due to gravity g (in meters/sec/sec) and it lands, again at ground level, R units away, after reaching a maximum height H. At what angle was the projectile fired? (Obviously the answer will involve some inverse trig function involving g, R and H.)

12. (10 points) A surface is given by $\varphi = k$, for a constant k, in spherical coordinates. (Recall that φ is the angle from the positive z-axis to a point.)

   (a) What kind of surface is this?

   It's a "half-cone" about the positive z-axis.

   (b) In the specific case that $\varphi = \pi/6$, convert to a recognizable form in rectangular coordinates, verifying your answer in part (a).

   We have

$$x = \rho \, \sin\phi \, \cos\theta = \rho\left(\frac{1}{2}\right)\cos\theta,$$

$$y = \rho \, \sin\phi \, \sin\theta = \rho\left(\frac{1}{2}\right)\sin\theta,$$

$$z = \rho \, \cos\phi = \rho\left(\frac{\sqrt{3}}{2}\right),$$

   we see that

$$x^2 + y^2 = \rho^2\left(\frac{1}{4}\right), \qquad z^2 = \rho^2\left(\frac{3}{4}\right),$$

   hence $z^2 = 3x^2 + 3y^2$. This is a cone; noting that $z \geq 0$ gives s the half cone,

$$z = \sqrt{3x^2 + 3y^2}.$$

⋆ ⋆ ⋆ ⋆ Extras ⋆ ⋆ ⋆ ⋆

A.) (⋆) Show that the curve $\mathbf{p}(t)$ given in problem #7 is contained in a single plane and give an equation for the plane.

B.) (⋆) For the vector valued function $\mathbf{r}(t)$ given in problem #2, find the unit binormal, $\mathbf{B}(t)$.

C.) (⋆) At what value(s) of t is the curvature greatest on the curve $\mathbf{r}(t)$ given in problem #2?

D.) (⋆) Find the point obtained when the point $(1, 2, 3)$ is reflected through the plane $x+2y+3z = 0$.

E.) ($\star\cdots\star$) Ask a question you wish I had asked and answer it. Points may vary.