Calc III — Spring 2006

MATH 223 — Name: ______________

Test #2

Instructions: Answer all problems correctly. Each numbered problem is worth 12 points. Each st$\star$rred problem is "extra credit" and each $\star$ is worth 8 points. Do as many problems as you wish, but no grade higher than 115 points (including a curve, if any) will be awarded on the test. You may leave unevaluated any integrals that need to be calculated in the course of solving these problems. In other cases, you may leave answers in unsimplified, "calculator-ready form" — you needn't rationalize denominators or do other trivial but time-consuming simplifications in your answers — unless some aspect of the answer needs simplification to be clear or unless otherwise specified. Calculators are not allowed.

1. Identify by name (e.g., ellipsoid, cone, paraboloid, etc.) the quadric surfaces defined by each of the following equations.

   (a) $x^2 - 2z^2 = y^2$

   (b) $4x - 2y^2 = z^2$

   (c) $x^2 - 2z^2 = y^2 + 4$

   (d) $x^2 - 2y^2 - z^2 = 4x + 4$

   (e) $x^2 - z = 4x + 4$

2. Convert the point given by the spherical coordinates $\rho = 2$, $\theta = \pi/3$, $\phi = 2\pi/3$ to rectangular coordinates. Convert the point in rectangular coordinates $(x, y, z) = (-3, -4, 5)$ to cylindrical coordinates. Simplify your answers as much as possible.

3. Describe in words the surfaces or figures whose equations are given.

   (a) $\rho = 6$

   (b) $r = 6$

   (c) $\theta = \pi/3$

   (d) $\phi = \pi/3$

   (e) $\phi = \pi/2$

   (f) $\phi = \pi$

   (g) $\rho = 6\cos\phi$

4. Find a parameterization for the curve formed by the intersection of the two surfaces

$$x^2 + 4y^2 = 9 \quad \text{and} \quad z = 1 + x^2.$$

5. Let $\mathbf{r}(t) = \langle 2t^2 - t,\ t^2,\ 3t + 2\rangle$ be a space curve. First, show that the point $(3, 1, -1)$ is on the curve. Next, find a parameterization for the line tangent to the curve at the point $(3, 1, -1)$.

6. Find the length of the curve $\mathbf{r}(t)$ given in problem #5 between the points $(0, 0, 2)$ and $(3, 1, -1)$.

7. Let $\mathbf{h}(t) = \langle 1 + 3t - 4\cos t,\ 2 + 4t + 3\cos t,\ 3 + 5\sin t\rangle$. Find the unit tangent vector to this curve when $t = \pi/2$.

8. Two particles travel along the following curves, parameterized by time t.

$$\mathbf{r}_1(t) = \langle t,\ t^2,\ t^3\rangle \quad \text{and} \quad \mathbf{r}_2(t) = \langle 2t - 6,\ 3t^2 - 8,\ t^2 - t - 10\rangle$$

   (a) Show that the particles do not collide.

   (b) Show that the curves do intersect. [It's possible that you could guess the point of intersection, but I won't accept that. The condition that the curves collide is that $\mathbf{r}_1(t) = \mathbf{r}_2(s)$ for some t and s. My advice is to solve the equations $x_1(t) = x_2(s)$, $y_1(t) = y_2(s)$ to find t and s, then verify that $z_1(t) = z_2(s)$ for at least one of these solutions.]

9. Prove the product rule for differentiating the dot product $\mathbf{u}(t) \cdot \mathbf{v}(t)$ of two vector-valued functions.

10. Suppose a particle moves along a path whose space curve is differentiable and that the particle is constrained to move on the surface of a sphere centered at the origin. Prove that the particle's tangent vector is perpendicular to its position vector.

11. Prove that for any twice-differentiable space curve $\mathbf{u}(t)$,

$$\frac{d}{dt}\left(\mathbf{u}'(t) \times \mathbf{u}(t)\right) = \mathbf{u}''(t) \times \mathbf{u}(t).$$

$\star\star\star\star$ Extras $\star\star\star\star$

A. ($\star$) Show that the curve in problem #5 lies entirely in a plane. Find an equation for the plane in standard form.

B. ($\star$) An ellipsoid is created by rotating the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ about the x-axis. Find an equation for the ellipsoid.

C. ($\star$) The *velocity* $\mathbf{v}(t)$ of a vector-valued function $\mathbf{r}(t)$ is defined by $\mathbf{v}(t) = \mathbf{r}'(t)$. The *speed* is then defined as $\|\mathbf{r}'(t)\|$. Show that the speed of the space curve given in problem #7 is constant.

D. ($\star$) Prove that if $\mathbf{r}(t)$ is differentiable, then

$$\frac{d}{dt}\|\mathbf{r}(t)\| = \frac{\mathbf{r}(t)}{\|\mathbf{r}(t)\|} \cdot \mathbf{r}'(t)$$

wherever $\mathbf{r}(t) \neq 0$.

E. ($\star$) Define what it means for a vector-valued function to be **smooth**. Which of the following curves are smooth? (Give correct justification for your answers, else no credit.)

(a) $\mathbf{u}(t) = \langle |t|, t^2, |t|^3 \rangle$

(b) $\mathbf{v}(t) = \langle t, t^2, t^3 \rangle$

(c) $\mathbf{w}(t) = \langle t^2, t^4, t^6 \rangle$

(d) $\mathbf{p}(t) = \langle t^2, t^3, t^4 \rangle$

F. ($\star\cdots\star$) Ask a question you wish I had asked and answer it. Points vary depending on the difficulty of the question (and the correctness of the solution). Very few points (if any) will be awarded for a problem that is essentially represented elsewhere on the test. (In other words, no repeats.)