Calc III — Fall 2003

MATH 223 — Name: ________________

Test #3

Instructions: Answer all problems correctly. Calculators are not allowed. Each st⋆rred problem is extra credit and each ⋆ is worth 10 points. I'll award a maximum of 120 points on this exam (including a curve, if any.)

1. (15 points) For each of the following, determine whether or not $f(x, y)$ has a limit at $(0, 0)$ *Justify your conclusions — no one-word answers.*

   a.) $f(x,y) = \dfrac{x^2 + 2xy + 4y^2}{x^2 + 4y^2}$

   No limit. Looking along $x = 0$ and $y = 0$, respectively, don't give sufficient info, since $f(0, y) \equiv 1$ for all $y \neq 0$ and $f(x, 0) \equiv 1$ for all $x \neq 0$, resp. But if we look along other lines, $y = mx$, we see that

$$f(x, mx) = \frac{x^2 + 2x(mx) + 4(mx)^2}{x^2 + 4(mx)^2} = \frac{1 + 2m + 4m^2}{1 + 4m^2} \qquad (\forall x \neq 0)$$

   which is not constant for all m (take $m = \pm 1$, for example).

   b.) $f(x,y) = \dfrac{x^3 + 2x^2y + 4y^3}{x^2 + 4y^2}$

   The limit is zero, as you suspect if you try various lines through the origin. A proof (of which many variations are possible) is as follows.

$$\begin{aligned}
|f(x,y) - 0| &= \frac{|x^3 + 2x^2y + 4y^3|}{x^2 + 4y^2} \\
&\leq \frac{|x|^3 + |2x^2y| + |4y^3|}{x^2 + 4y^2} \qquad \text{(by the triangle inequality)} \\
&= \frac{x^2|x| + 2x^2|y| + 4y^2|y|}{x^2 + 4y^2} \\
&\leq \frac{(x^2 + 4y^2)|x| + 2(x^2 + 4y^2)|y| + (x^2 + 4y^2)|y|}{x^2 + 4y^2} \\
&= |x| + 3|y|,
\end{aligned}$$

   which goes to zero. So the limit is squeezed to zero.

c.) $f(x,y) = \dfrac{x^4 + 2x^2y^2 + 4y^4}{x^2 + 4y^4}$

This one has no limit. Letting $x = 0$ and $y = 0$ easily gives two different limits.

2. (15 points) Let $f(x,y) = x^2 + y^2 + x^2 y$.

   (a) Find the partial derivatives, $f_x(x,y)$ and $f_y(x,y)$.

$$f_x(x,y) = 2x + 2xy \quad \text{and} \quad f_y(x,y) = 2y + x^2.$$

   (b) Evaluate the above at the point $P = (2,1)$.

$$f_x(2,1) = 8 \quad \text{and} \quad f_y(2,1) = 6.$$

   (c) Find the second partials, $f_{xx}(x,y), f_{xy}(x,y), f_{yx}(x,y), f_{yy}(x,y)$.

$$f_{xx}(x,y) = 2 + 2y, \quad f_{xy}(x,y) = f_{yx}(x,y) = 2x, \quad f_{yy}(x,y) = 2.$$

3. (10 points) The point $(1,-1,1)$ lies on the surface defined by $x^2 - xz + yz^3 + 1 = 0$. Find $\dfrac{\partial z}{\partial x}$ at that point.

First it must be noted that the point $(1,-1,1)$ does indeed lie on the surface given, by direct substitution. So we can implicitly differentiate to get

$$2x - z - x\frac{\partial z}{\partial x} + 3yz^2\frac{\partial z}{\partial x} = 0,$$

which we solve for $\dfrac{\partial z}{\partial x}$ to get

$$\frac{\partial z}{\partial x} = \frac{z - 2x}{3yz^2 - x}.$$

Plugging in the point gives

$$\frac{\partial z}{\partial x} = \frac{1 - 2(1)}{3(-1)1^2 - 1} = \frac{1}{4}.$$

You could also use a formula derived in class. Let the left-hand side of the surface's defining equation be $G(x,y,z)$, so the surface is given by $G(x,y,z) = 0$. (Any rearrangement of the form $G(x,y,z) = \text{constant}$ will do.) Then use the formula $\dfrac{\partial z}{\partial x} = -G_x/G_z$.

4. (10 points) Referring to problem #2, find the linear approximation to the function $f(x,y)$ at the point P.

The linearization is the equation of the tangent plane,

$$\begin{aligned} L(x,y) &= f(2,1) + (x-2)f_x(2,1) + (y-1)f_y(2,1) \\ &= 9 + 8(x-2) + 6(y-1), \end{aligned}$$

or if you prefer, $L(x,y) = 8x + 6y - 13$.

5. (10 points) In problem #2, in the direction of which unit vector is there the greatest rate of increase of $f(x, y)$ at the point P?

   The rate of greatest increase is in the direction of the gradient, $\nabla f(2,1) = \langle 8, 6\rangle$, so the desired unit vector is $\langle 4/5, 3/5\rangle$.

6. (10 points) In problem #2, what is the rate of change of $f(x, y)$ at the point P as one begins to move directly toward the origin?

   We are being asked for the directional derivative $D_{\hat{\mathbf{u}}}f(2,1)$, where $\hat{\mathbf{u}}$ is the unit vector pointing from $(2,1)$ to $(0,0)$, i.e., $\hat{\mathbf{u}} = \langle -2\sqrt{5}, -1/\sqrt{5}\rangle$. So

$$\begin{aligned} D_{\hat{\mathbf{u}}}f(2,1) &= \nabla f(2,1) \cdot \hat{\mathbf{u}} \\ &= \langle 8, 6\rangle \cdot \langle -2\sqrt{5}, -1/\sqrt{5}\rangle \\ &= -22/\sqrt{5}. \end{aligned}$$

7. (10 points) In problem #2, let $u(r,t) = r^2 - rt$ and $v(r,t) = 2r + 3t$. Let $g(r,t) = f(u(r,t), v(r,t))$. Use the chain rule to find $g_t(1,-1)$.

The chain rule implies (writing in one of the terrible chain rule notations) that

$$\frac{\partial g}{\partial t} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial t},$$

which translates to

$$g_t(r,t) = f_x(u(r,t), v(r,t))u_t(r,t) + f_y(u(r,t), v(r,t))v_t(r,t).$$

You'll notice that

$$u(1,-1) = 2 \quad \text{and} \quad v(1,-1) = -1,$$

so you already have f_x and f_y and so you can get

$$f_x(2,-1) = 0, \quad f_y(2,-1) = 2.$$

Now just use

$$u_t(r,t) = -r \quad \text{and} \quad v_t(r,t) = 3$$

to get

$$u_t(1,-1) = -1 \quad \text{and} \quad v_t(1,-1) = 3$$

and finally,

$$g_t(1,-1) = 0 \cdot (-1) + 2 \cdot 3 = 6.$$

8. (15 points) For the function $f(x,y)$ in problem #2, find all the critical points of $f(x,y)$ and classify the behavior at each of these (as local max, saddle point, etc.).

Our critical points are the solutions to

$$f_x(x,y) = 2x + 2xy = 0 \quad \text{and} \quad f_y(x,y) = 2y + x^2 = 0,$$

the first of which gives either $x = 0$ or $y = -1$. If $x = 0$ then the second equation gives $y = 0$, so $(0,0)$ is a critical point; if $y = -1$ we get $x^2 = 2$ so $(\pm\sqrt{2}, -1)$ are two more, and these three are the only ones. We also have

$$f_{xx}(x,y) = 2 + 2y, \quad f_{xy}(x,y) = f_{yx}(x,y) = 2x, \quad f_{yy}(x,y) = 2,$$

so

$$\mathcal{D} := f_{xx}f_{yy} - f_{xy}^2 = (2+2y)(2) - 4x^2.$$

Clearly, $\mathcal{D} = 4 > 0$ at $(0,0)$, and since $f_{xx} > 0$ there (as is f_{yy}), there is a relative minimum there. At $(\pm\sqrt{2}, -1)$, $\mathcal{D} = -8 < 0$, so each of these two c.p. is a saddle point.

9. (15 points) For the function $f(x, y)$ in problem #2, find the absolute maximum and minimum value of $f(x, y)$ on the closed triangular region having vertices $(2, 1)$, $(-1, 1)$, $(2, -2)$. (There are a few square roots involved in some of the evaluations, but don't panic! They are relatively easily approximated. Also, not all the critical points of the function are inside the region, so don't evaluate any that aren't necessary.)

We already have the c.p. from the previous problem, and moreover, two of these are saddle points which we may ignore! To evaluate things on the boundary of the triangle we can form three functions, which I'll denote as g_1, g_2 and g_3, which will describe $f(x, y)$ when restricted to the segment along the top of the top ($y = 1$) of the triangle, along the vertical side ($x = 2$) and along the remaining side ($y = -x$). First,

$$g_1(x) = f(x, 1) = 1 + 2x^2, \quad -1 \leq x \leq 2,$$

which has a minimum a c.p. at $x = 0$ (clearly a local minimum for g_1) and we get the corresponding c.p. $(0, 1)$ of f. Next,

$$g_2(y) = f(2, y) = (y + 2)^2, \quad -2 \leq y \leq 1,$$

which has a c.p. at $y = -2$. But this corresponds to the "corner" point $(2, -2)$, which we'll deal with separately. Finally,

$$g_3(x) = f(x, -x) = -x^3 - 2x^2, \quad -1 \leq x \leq 2,$$

which has c.p. at $x = 0$ and $x = -4/3$. The latter is not in our region of interest, and the other is the origin, which we are already aware of as a c.p. So, we tabulate the function at these points and at the vertices of the triangle and get

(x, y)	$f(x, y)$	
$(0, 0)$	$f(0, 0) = 0$	(min)
$(0, 1)$	$f(0, 1) = 1$	
$(-1, 1)$	$f(-1, 1) = 3$	
$(2, 1)$	$f(2, 1) = 9$	(max)
$(2, -2)$	$f(2, -2) = 0$	(min)

$\star\star\star\star$ EXTRAS $\star\star\star\star$

A.) ($\star$) Discuss the limits $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+xy+y^2}$ and $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+3xy+y^2}$.

B.) ($\star$) For the function $f(x,y)$ in problem #2, verify the differentiability of f at P by finding functions ε_1 and ε_2 that satisfy the criterion in the definition of differentiability.

C.) ($\star$) If $f(x,y) = \dfrac{\sin(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}$, find $f_y(0,0)$.

D.) ($\star$) Give an explicit formula for a function $f(x,y)$ that has a saddle point at $(0,0)$ but for which the second derivative test is inconclusive.

E.) ($\star\cdots\star$) Ask a question you wish I had asked and answer it. Points may vary.