Calc III Spring 2006
MATH 223 Name:

Test #3

Instructions: Answer all problems correctly. Each numbered problem is worth 12 points. Each st$\star$rred problem is "extra credit" and each $\star$ is worth 8 points. Do as many problems as you wish, but no grade higher than 115 points (including a curve, if any) will be awarded on the test. Calculators are not allowed. *Clarity counts, so be clear. Points will be lost if I have to guess what you're doing. This is especially true when you are asked to prove something.*

1. (15 points) Consider the curve $\mathbf{r}(t) = \langle 2t^2 - t,\ t^2,\ 3t + 2\rangle$. $\boxed{15}$

   (a) Calculate the curvature $\kappa(t)$ at each t.

   (b) Calculate the tangential and normal components, $a_T(t)$ and $a_N(t)$, of the acceleration vector.

   (c) What is the maximum curvature and for what t is it attained?

2. (8 points) Consider the ellipse given by $x(t) = a\cos t, \quad y(t) = b\sin t$ and assume $a > b > 0$. $\boxed{23}$

   (a) Calculate the *radius* of curvature at each t.

   (b) Find the maximum and minimum values of the radius of curvature.

3. (15 points) Begin with the typical, basic assumptions of projectile motion near the surface of the earth, namely that $x''(t) = 0$ and $y''(t) = -g$, where g is a constant (the constant downward acceleration due to gravity). Thus we have $\mathbf{r}''(t) = \langle 0,\ -g\rangle$. Assume that a projectile is launched from an initial position of (x_0, y_0) with an initial speed v_0 and an initial angle θ relative to the positive x-axis. $\boxed{38}$

   (a) By integrating once, derive the equation for the velocity,

   $$\mathbf{r}'(t) = \mathbf{v}(t) = \langle v_0\cos\theta,\ v_0\sin\theta - gt\rangle.$$

   (b) By integrating once more, derive the equation for the position,

   $$\mathbf{r}(t) = \langle x_0 + (v_0\cos\theta)t,\ y_0 + (v_0\sin\theta)t - gt^2/2\rangle.$$

   (c) Show that the maximum height attained by the projectile is

   $$y_0 + \frac{v_0^2\sin^2\theta}{2g}\ .$$

   (This assumes that θ is in either quadrant I or II, of course, i.e., that we aren't shooting the projectile toward the ground at $t = 0$.) Show all steps.

4. (8 points) Find and sketch the domain of the functions below. $\boxed{46}$

   (a) $f(x, y) = \sqrt{x} - \sqrt{y-1}$.

   (b) $f(x, y) = \ln(4 - x^2 - y^2)$.

5. (10 points) For each of the following, prove that $f(x, y)$ does not have a limit at $(0, 0)$. $\boxed{56}$

   (a) $f(x, y) = \dfrac{2x^2 - xy + 3y^2}{x^2 + 2y^2}$

   (b) $f(x, y) = \dfrac{x^4 + 3x^2y^2 + 5y^4}{x^2 + 4y^4}$

6. (10 points) Prove that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^3 - x^2y + 3y^3}{x^2 + y^2} = 0$. (I recommend "squeezing" the results out of it, but there are other ways. Take care to justify any inequality you use.) $\boxed{66}$

7. (10 points) Let $f(x, y) = \sin(5x - 7y) + x^2y^3$. $\boxed{76}$

   (a) Find the partial derivatives, $f_x(x, y)$ and $f_y(x, y)$.

   (b) Find the second partials, $f_{xx}(x, y), f_{xy}(x, y), f_{yx}(x, y), f_{yy}(x, y)$.

8. (10 points) The point $(1, 2, 3)$ lies on the surface defined by $\boxed{86}$

$$y^2 + 3xyz - x^2z^3 = -5.$$

   Find $\dfrac{\partial z}{\partial y}$ at that point.

9. (5 points) Show that the function $\boxed{91}$

$$z = e^{-5x} \sin 5y$$

   solves *Laplace's equation*,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

10. (10 points) Write concisely, completely and clearly the *definition of differentiability* for a real-valued function of two variables. $\boxed{101}$

11. (10 points) Prove that the function $f(x, y) = 3xy + x^2$ is differentiable at the point $P = (1, 2)$. (Reminder: find suitable ε_1 and ε_2 and show that they have the necessary properties.) $\boxed{111}$

12. (10 points) Let the function f and the point P be as in the previous problem. $\boxed{121}$

    (a) Find an equation for the tangent plane to $z = f(x, y)$ at P.

    (b) Find an expression for the linearization of f at P.

$\star\star\star\star$ Extras $\star\star\star\star$

A. ($\star\star$) Referring to problem #3, let $v_0 \neq 0$, assume that $0° < \theta < 90°$ and let $(x_0, y_0) = (0, 0)$. The *range* of the projectile is the value of $x(t)$ when the object hits the ground. Find the range as a function of v_0 and θ. Then, show that for a fixed value of v_0, the greatest value for the range occurs when $\theta = 45°$.

B. ($\star$) Show that the curve in problem #1 lies entirely in a plane. Find an equation for the plane in standard form.

C. ($\star$) Repeat problem #6, but first change the denominator of the function to $2x^2 + 3y^2$.

D. ($\star$) State clearly and completely **Clairaut's Theorem** concerning mixed partials.

E. ($\star$) Find $f_y(0, 0)$ if $f(x, y) = \sqrt[5]{x^5 + y^5}$. Show all steps, which count.

F. ($\star$) Give an example of a function $f(x, y)$ whose mixed partial derivatives f_{xy} and f_{yx} exist at a point but are not equal there.

G. ($\star$) If you're standing on the surface given by $z = f(x, y)$ at the point $(a, b, f(a, b))$, how "steep" is the surface there? Answer this by finding the angle that the tangent plane makes with respect to the plane $z = 0$. (We'll talk more about this idea soon, when we discuss the **gradient** of a function.)

H. ($\star$) Suppose that the functions $f(x, y, z), u(s, t), v(s, t)$ and $w(s, t)$ are all differentiable. Let

$$z = f(u(s, t),\ v(s, t),\ w(s, t)).$$

Use the chain rule to write the formula for $\dfrac{\partial z}{\partial t}$.

I. ($\star \cdots \star$) Ask a question you wish I had asked and answer it. Points vary depending on the difficulty of the question (and the correctness of the solution). Very few points (if any) will be awarded for a problem that is essentially represented elsewhere on the test. (In other words, no repeats.)