Calc III | Spring 2013
MATH 223 | Name:

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Test #3

Instructions: Answer all problems correctly. Each st⋆rred problem is "extra credit" and each ⋆ is worth 5 points. Do as many extra-credit problems as you wish, but no grade higher than 110 points (including a curve, if any) will be awarded on the test. You may leave answers in unsimplified, "calculator-ready form" — you needn't rationalize denominators or do other trivial but time-consuming simplifications in your answers — unless some aspect of the answer needs simplification to be clear or unless otherwise specified. Calculators are not allowed.

1. (5 points) Let $f(x, y)$ be defined on a set $D \subset \mathbb{R}^2$. Write the precise mathematical *definition** of the statement

$$\lim_{(x,y)\to(a,b)} f(x, y) = L.$$

   (*Hint: it had better have an epsilon in it. No points will be given for vague impressions or intuitive notions of what limits are like. Just the facts, please. Keep in mind that your text's definition allows limits at boundary points.)

2. (3 points) State precisely *Clairaut's Theorem.*

3. (5 points) State the precise definition of what it means for $f(x, y)$ to be differentiable at (a, b).

4. (8 points) Use the definition of differentiability to prove that the function

$$f(x, y) = xy - 2x$$

   is differentiable at $(1, 1)$.

5. (6 points each) For each of the following, state whether or not $f(x, y)$ has a limit at $(0, 0)$ and calculate the limit if it does exist. Full credit will be given only is answers are justified; half-credit will be given for correct answers lacking an explanation.

   (a) $f(x, y) = \dfrac{(x+1)^3 + y^3}{(x+1)^2 + y^2}$

   (b) $f(x, y) = \dfrac{x^2 - 2\,x\,y + y^2}{x + y}$

   (c) $f(x, y) = \dfrac{2\,x\,y}{x^2 + 2\,y^2}$

6. (8 points) Prove that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^3 - x^2y + 3y^3}{x^2 + y^2} = 0$ by "squeezing" it.

7. (9 points) Let $f(x, y) = (xy - 2x)^3$. Find these three second partials: $f_{xx}(x, y)$, $f_{yx}(x, y)$, and $f_{xy}(x, y)$. (You can skip steps if you know how. Be careful; you might be able to re-use some of the work in problems #14 and ⋆B)

8. Consider the surface defined by $x^2z^2 - ze^{-xyz} + y = 6$.

   (a) (1 point) Verify that the point $(1, 0, -2)$ lies on the surface.

   (b) (9 points) Find $\dfrac{\partial z}{\partial y}$ at that point.

9. (4 points) Write the chain rule for

$$\frac{\partial}{\partial t} f(x, y, z)$$

   when

$$x = u(s, t), \quad y = v(s, t), \quad z = w(s, t),$$

   and u, v, w are differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}$.

10. (8 points) <u>Use the chain rule</u> to evaluate $\partial z/\partial s$ and $\partial z/\partial t$, where

$$z = e^{xy} \tan y, \quad x = s + 2t, \quad y = s/t.$$

11. (8 points) Let $f(x, y)$ be a differentiable function of x and y and let $g(x) = f(x, x^2)$. Use the chain rule to write a formula for $g'(x)$.

12. (8 points) At the moment its radius is 10 cm and its height is 5 cm, the radius of a right circular cone is increasing at a rate of 2 cm/min, while its height is decreasing at 3 cm/min. How fast is the volume of the cone changing at that instant? (If you do not remember that the volume of a cone is given by $V = \pi R^2 H/3$, please let me know, so I can deduct 3 points.)

13. (4 points) Give the definition (involving the limit) for the directional derivative $D_{\hat{u}} f(x_0, y_0)$ of f at (x_0, y_0) in the direction of a unit vector $\hat{u} = \langle u_1, u_2 \rangle$.

14. (10 points) Let f be the function given in problem #7.

    (a) Calculate the gradient $\vec{\nabla} f(x, y)$ at the point $(1, 1)$.

    (b) Find the directional derivative of f at $(1, 1)$ in the direction of the point $(4, 5)$.

15. (8 points) A weird oven is a cube that can be taken to occupy the region $[0, 4]^3$ in $\mathbb{R}^3$, the measurements being in meters. But all you need to know is that the temperature T (in degrees Celsius) at each point (x, y, z) in the oven is given by $T(x, y, z) = xy^2z^3$. At the point $(1, 2, 3)$, in what direction does the temperature decrease the fastest?

$\star\star\star\star$ EXTRAS $\star\star\star\star$

(A) ($\star$) Describe and sketch the domain of the functions below.

(a) $f(x,y) = \ln(4 - x - y)$.

(b) $f(x,y) = (\ln x)\sqrt{4 - x^2 - y^2}$.

(B) ($\star$) Write an equation for the tangent plane to the surface $z = f(x,y)$ at the point where $(x,y) = (1,1)$, and where f is the function given in problem #7.

(C) ($\star$) If $z = f(x,y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, prove that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

(Assume all the functions are differentiable.)

(D) ($\star\cdots\star$) Ask a question you wish I had asked and answer it. Points vary depending on the difficulty of the question (and the correctness of the solution). Very few points (if any) will be awarded for a problem that is essentially represented elsewhere on the test. (In other words, no repeats.)