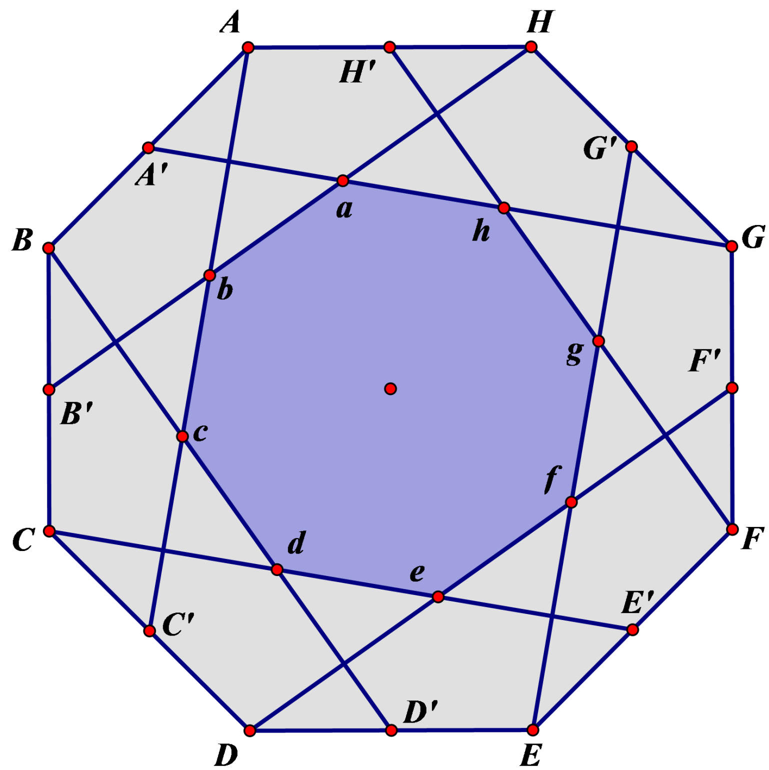
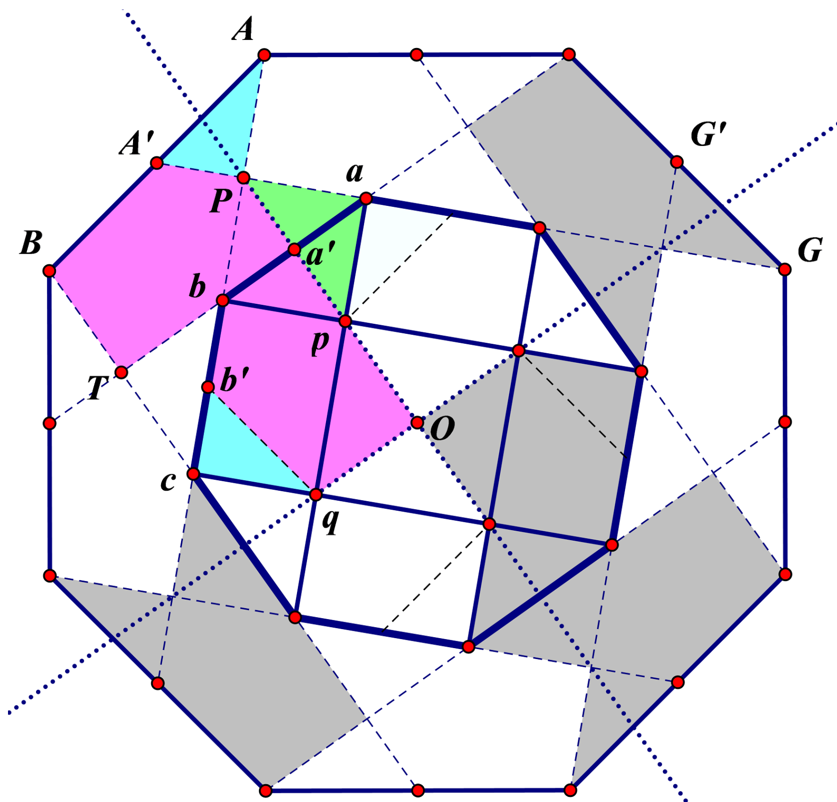
Proposed by Rick Mabry, Dept. of Mathematics, LSU Shreveport, Shreveport, LA 71115-2399, Richard.Mabry@LSUS.edu

Let A' be the midpoint of segment $[AB]$ on a regular octagon $[ABCDEFGH]$ and likewise define the other seven midpoints B' through H' cyclically. Let a denote the intersection of segments $[A'G]$ and $[B'H]$ and define b through h cyclically. Prove that the inner regular octagon $[abcdefgh]$ has one-third the area of the outer octagon $[ABCDEFGH]$.



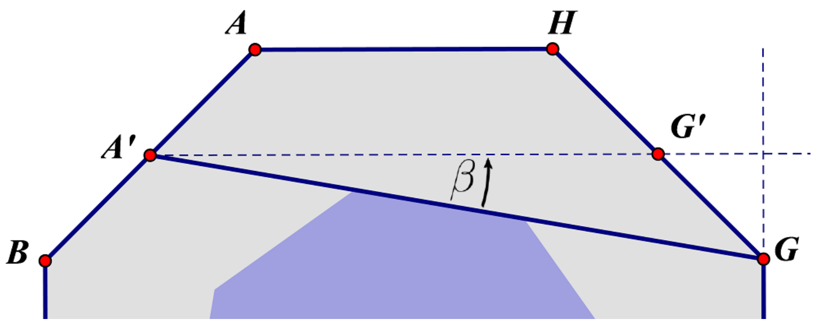
Solution. One can arrive at an easy proof algebraically with vectors or complex numbers. The following image shows a dissection that yields a proof.



Eight congruent copies of polygon $M_0 = [AA'BTaP]$ form the outer ring—the region outside the inner octagon. (These copies are obtained by rotating M_0 by multiples of $45°$ about the octagons' center O.) Dividing M_0 into the three pieces $M_1 = [PA'BTa']$ (pink), $M_2 = [a'aP]$ (green), and $M_3 = [AA'P]$ (cyan), we see that polygon $m_0 = [Opabb'q]$ within the inner octagon appears to be comprised of congruent copies of M_1, M_2, and M_3, namely, $m_1 = [bb'qOa']$ (pink), $m_2 = [a'pa]$ (green), and $m_3 = [qb'c]$ (cyan). If this is true, then since the inner ring consists of four congruent copies of m_0, it follows that the inner polygon has area equal to $4/(4+8) = 1/3$ times the area of the outer polygon.

It would be nice to have a completely visual proof of the congruence suggested in the previous paragraph—let's hope a reader submits one. Meanwhile, we can prove the congruence using some simple trigonometry, which at least confirms our claim.

We assume throughout that $AB = 1$.



Let $\beta = \angle G'A'G$ and let $\alpha = \angle AA'G$. By considering the added lines parallel to segments $[AH]$ and $[GF]$ we can deduce that

$$\tan\beta = \frac{\frac{1}{2}\sqrt{1/2}}{\frac{3}{2}\sqrt{1/2}+1} = \frac{1}{3+2\sqrt{2}} = 3-2\sqrt{2}$$

and from there we find (after some simplification) that

$$\tan\alpha = \tan(45^\circ+\beta) = \frac{1+\tan\beta}{1-\tan\beta} = \sqrt{2}.$$

Back to our congruences, we now have that $\tan\angle AA'P = \sqrt{2}$. Letting $x = ab$ it is easy to see that $\tan\angle qb'c = (x/\sqrt{2})/(x/2) = \sqrt{2}$ and it follows that $\angle AA'P \cong \angle qb'c$. That implies via supplements that $\angle PA'B \cong \angle bb'q$. Using fairly obvious relationships and summing angles in M_1 and m_1 we obtain that $\angle A'BT \cong \angle b'qO$, so we now can quickly see that $M_1 \cong m_1$. It was already clear at a glance that $M_2 \cong m_2$. Finally, our observations about angles already tell us that M_3 and m_3 are similar, but now the shared side of M_3 and m_3 gives $qb' = A'B = AA'$, hence $M_3 \cong m_3$. ✓