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Proposed Problem. Let $f(x)$ and $g(y)$ be twice continuously differentiable functions defined in a neighborhood of zero, and assume that

1. $f(0) = 1$ and $g(0) = 0$,
2. $f'(0) = 0$ and $g'(0) = 0$,
3. $f''(0) < 0$ and $g''(0) > 0$.

Consider the curves $x = g(y)$ and $y = rf(x/r)$. For sufficiently small $r > 0$, show that the curves have an intersection (x_r, y_r) in the first quadrant of smallest x-value. (So, loosely speaking, there is a "first" intersection point. In general, there may be infinitely many such intersection points in the first quadrant even for arbitrarily small r, but there is one nearest to the y-axis.) Let $(t_r, 0)$ denote the x-intercept of the line passing through the points $(0, r)$ and (x_r, y_r).

a.) Show that $\lim_{r\to 0^+} t_r$ necessarily exists, and find the limit. (This is a generalization of Problem #5 in "Which Way did the Bicycle Go?" by Konhauser, Velleman, and Wagon, in which $f(x) = \sqrt{1-x^2}$ and $g(y) = 1 - \sqrt{1-y^2}$.)

b.) Is the condition of the continuity of f'' and g'' necessary?

Solution.

a.) The limit exists and equals $\dfrac{-4}{g''(0)f''(0)}$.

Proof. First we show the existence of (x_r, y_r) for r sufficiently small. Since $f''(0) < 0$ and f'' is continuous on an interval containing zero, there exists $a > 0$ such that f is defined on $[0, a]$, and for all $x \in (0, a]$, $0 < f(x) \le 1$, $f'(x) < 0$, and $f''(x) < 0$. We shall therefore assume that f is defined only on $[0, a]$. Similarly, since $g''(0) > 0$ and g' is continuous at the origin, there exists c such that $0 < c < f(a)$ with $0 < g'(z) < a/f(a)$ for all $z \in (0, c)$. The Mean Value Theorem implies that $0 < g(z) < (a/f(a))z$ for all $z \in (0, c)$. Consider the quadrilateral Q with vertices $(0, 0)$, $(0, c)$, $(g(c), c)$, and $(g(c), f(a)g(c)/a)$. The graph of $x = g(y)$ for $0 \leq y \leq c$ is a continuous, strictly increasing function from the origin to the horizontal side of Q. For any r such that $0 < r < g(c)/a$, the graph of $y = rf(x/r)$ for $0 \leq x \leq ar$ is a continuous, strictly decreasing function which joins the horizontal side of Q (if $c < r$) or the vertical side of Q which contains the origin (if $r < c$) to the oblique side of Q– this graph is just a "scaled-down" version of the graph of $y = f(x)$ for $0 \leq x \leq a$. Thus it is obvious that the intersection point (x_r, y_r) exists and is unique in the interval $0 \leq x \leq ar$ for all $0 < r < g(c)/a$ (see the diagram at the end of this note for the case $r < c$). For those who require a rigorous proof of this fact using the Intermediate Value Theorem, we include it at the end of part (a).

Since the curve $y = rf(x/r)$ is strictly decreasing for $0 \leq x \leq ar$, for $0 < r < g(c)/a$ it is clear that $y_r < r$ and trivially $x_r < ar$, that is, $x_r/r < a$.

Using Taylor's theorem, there exists $\zeta_r \in (0, y_r)$ such that

$$\begin{aligned} \frac{x_r}{r} &= \frac{g(y_r) - g(0)}{r} \\ &= \frac{1}{2r} y_r^2 g''(\zeta_r) \qquad (1) \\ &\to 0 \quad \text{as } r \to 0, \end{aligned}$$

since $y_r < r$ and g'' is continuous at the origin, hence bounded as $r \to 0$. Since $y_r \to 0$ we see that $\zeta_r \to 0$ as $r \to 0$.

Similarly, using Taylor again along with (1), there exists $\xi_r \in (0, x_r/r)$ such that

$$\frac{y_r}{r} - 1 = f\left(\frac{x_r}{r}\right) - f(0)$$

$$= \frac{1}{2}\left(\frac{x_r}{r}\right)^2 f''(\xi_r) \tag{2}$$

$$\to 0 \quad \text{as } r \to 0, \tag{3}$$

where $\xi_r \to 0$ as $r \to 0$.

An easy calculation of t_r in terms of x_r and y_r combined with (2), (1), and (3) (in that order) along with the continuity of f'' and g'' at the origin, yields

$$\begin{aligned} t_r &= \frac{-rx_r}{y_r - r} \\ &= \frac{-x_r}{f(x_r/r) - f(0)} \\ &= \frac{-2}{x_r f''(\xi_r)/r^2} \\ &= \frac{-4}{(y_r/r)^2 g''(\zeta_r) f''(\xi_r)} \\ &\to \frac{-4}{g''(0)f''(0)} \end{aligned}$$

as $r \to 0$, completing the proof.

Now we provide a rigorous argument verifying the existence of the intersection point (x_r, y_r) for r small enough. Choose a and c as above, recall that $0 < g(z) < (a/f(a))z$ for all $z \in (0, c)$, and look at the function

$$h_r(y) \equiv y - rf(g(y)/r).$$

Observe that $h_r(0) = -r < 0$, so if we can find some $z_r > 0$ for which $h_r(z_r) > 0$, we will be done by the Intermediate Value Theorem. Let $r \in (0, g(c)/a)$. Since g is increasing on

$(0, c)$, it is one-to-one there. Since $0 < ar < g(c)$, there must be some $z_r \in (0, c)$ for which $g(z_r) = ar$. Then

$$\begin{aligned} h_r(z_r) &= z_r - rf(g(z_r)/r) \\ &= z_r - rf(a) \\ &> 0, \end{aligned}$$

since $f(a) < az_r/g(z_r) = az_r/(ra) = z_r/r$. Thus there exists $y_r \in (0, z_r)$ for which $h_r(y_r) = 0$ and we let $x_r = g(y_r)$, completing the existence proof.

The point of intersection is unique in the interval $[0, ar]$ by the strict monotonicity of the two functions on this interval. ✡

b.) The requirement of continuity of the second derivatives is unnecessary. In fact, the result holds so long as the first derivatives exist on an interval about 0, and even if the second derivatives do not exist at any points other than zero. The result can then be proven using the following

Theorem. Let f be differentiable on an open interval J containing a and suppose that $f''(a)$ exists and is not zero. For each $x \in J \setminus \{a\}$, let c_x denote any number between a and x for which $f(x) = f(a) + (x - a)f'(c_x)$. Then

$$\lim_{x \to a} \frac{c_x - a}{x - a} = \frac{1}{2}.$$

For the proof of this theorem, see the note "On the Meanness of the Mean Value in the Mean Value Theorem" in this *Monthly*.

Now, even without the above theorem, the result (1) holds, since by Taylor's theorem, there exists $\zeta_r \in (0, y_r)$ such that

$$\frac{x_r}{r} = \frac{g(y_r) - g(0)}{r}$$

$$= \frac{1}{r} y_r g'(\zeta_r)$$

$$\to 0 \quad \text{as } r \to 0,$$

since $y_r < r$ and g' is continuous at 0. (Note that we did not explicitly say that g' is continuous at 0, but it follows because g' is defined in a neighborhood of 0, $g'(0) = 0$, and $g''(0)$ exists.)

To obtain (3), we likewise have $\xi_r \in (0, x_r/r)$ for which

$$\frac{y_r}{r} - 1 = f\left(\frac{x_r}{r}\right) - f(0)$$

$$= \frac{x_r}{r} f'(\xi_r)$$

$$\to 0 \quad \text{as } r \to 0.$$

Finally, the above-mentioned Theorem is used to prove the final result, by implying the following two limits.

$$\lim_{r \to 0} \frac{\zeta_r}{y_r} = \frac{1}{2} \qquad \text{and} \qquad \lim_{r \to 0} \frac{\xi_r}{(x_r/r)} = \frac{1}{2}.$$

(The ζ_r and ξ_r below are as in the two preceding paragraphs.)

We have, finally, that

$$x_r = g(y_r) - g(0) = y_r g'(\zeta_r) = r^2 \left(\frac{y_r}{r}\right)^2 \frac{g'(\zeta_r)}{\zeta_r} \frac{\zeta_r}{y_r},$$

and so

$$t_r = \frac{-r\ x_r}{y_r - r} = \frac{-x_r}{f(x_r/r) - f(0)}$$

$$= \frac{-x_r}{\dfrac{x_r}{r} f'(\xi_r)} = \frac{-\dfrac{x_r}{\xi_r}}{\dfrac{x_r}{r}\,\dfrac{f'(\xi_r)}{\xi_r}}$$

$$= \frac{-r^2 \left(\dfrac{x_r}{r\xi_r}\right)}{r^2 \left(\dfrac{y_r}{r}\right)^2 \dfrac{g'(\zeta_r)}{\zeta_r}\,\dfrac{\zeta_r}{y_r}\,\dfrac{f'(\xi_r)}{\xi_r}}$$

$$\to \frac{-4}{g''(0) f''(0)}, \quad \text{as } r \to 0,$$

and we are done.