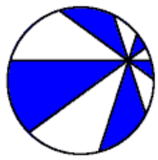
Paul Deiermann and Rick Mabry, “The center of a sliced pizza”, *Mathematics Magazine*, **68**, no. 4 (1995), 312-315. (Printed here with minor additions.)



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1457. *Proposed by Larry Carter, IBM Watson Research Center, Yorktown Heights, New York, John Duncan, University of Arkansas, Fayetteville, Arkansas, and Stan Wagon, Macalester College, St. Paul, Minnesota.*

a. For a point P inside a circle draw three chords through P making six $60°$ angles at P and form two regions by coloring the six “pizza slices” alternately black and white. Prove that the region containing the center has the larger area.

b.* Prove that if five chords make ten $36°$ angles at P, then the region containing the center has the lesser area.

Solution by Paul Deiermann and Rick Mabry, Louisiana State University in Shreveport, Shreveport, LA 71115-2399.

a. Our pizza is the unit disk. Without loss of generality, let P have polar coördinates (r, θ), with $0 \leq \theta < \pi/3$, and let one of the cuts be parallel to the x-axis. (Any other configuration can be rotated into one like this.) We prove the result by showing that the region containing the origin, call it $\mathcal{R}(|||)$ (the shaded region in figure 1), comprises no less than half the area of the pizza. (We will not always explicitly distinguish regions from their areas, it being clear from the context.)

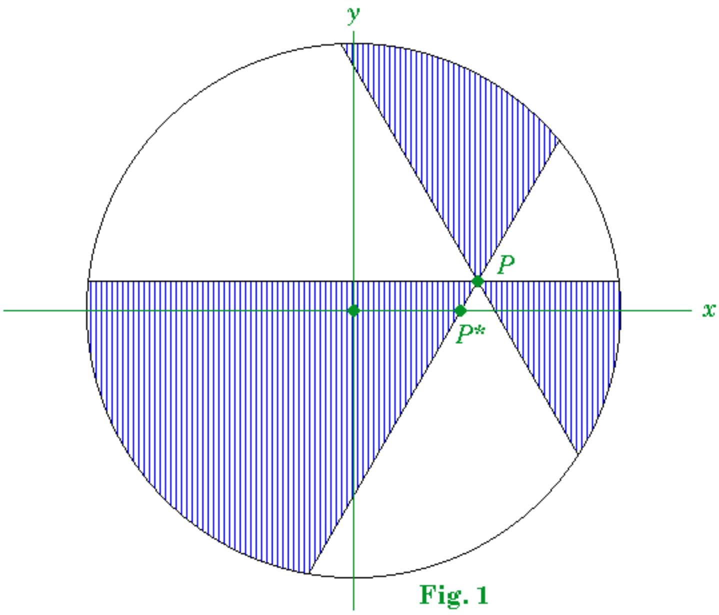

Fig. 1

Let P^* denote the intersection of the x-axis with the cut in the direction of $\pi/3$. Now take a fresh pizza and make cuts parallel to those on the first, but through P^*. Then $\mathcal{R}(\equiv)$, the region on the second pizza corresponding to $\mathcal{R}(|||)$, is exactly one half of the pizza. ("Pie over two." See figure 2.) If $\mathcal{R}(|||)$ is bigger than or equal to $\mathcal{R}(\equiv)$, we're done.

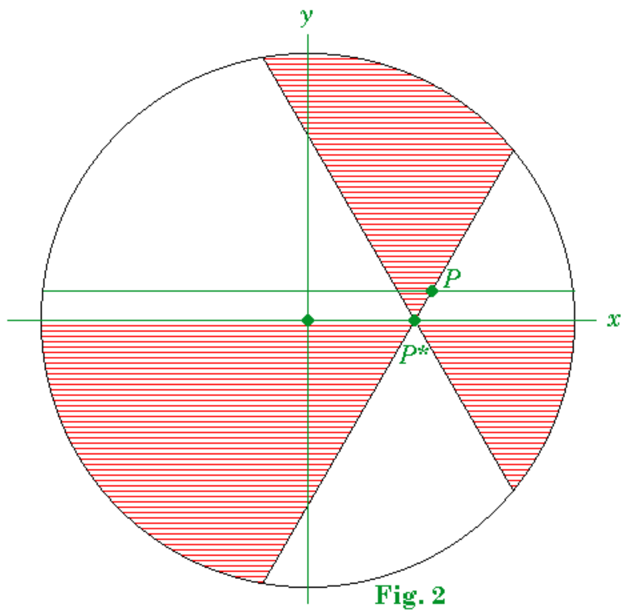

Fig. 2

Overlay the two pizzas, as in figure 3. (This picture <u>is</u> the proof.) We disregard the doubly shaded regions, since they occur in both pizzas, and compare what is left, the two "strips," $ABCD$ and $A'B'C'D'$, minus their intersection. But the intersection, being common to both, can be ignored and we are left comparing the filled strips. These strips are of equal thickness, but $ABCD$ is clearly the larger, since $A'B'C'D'$ sits at a greater distance from the center.

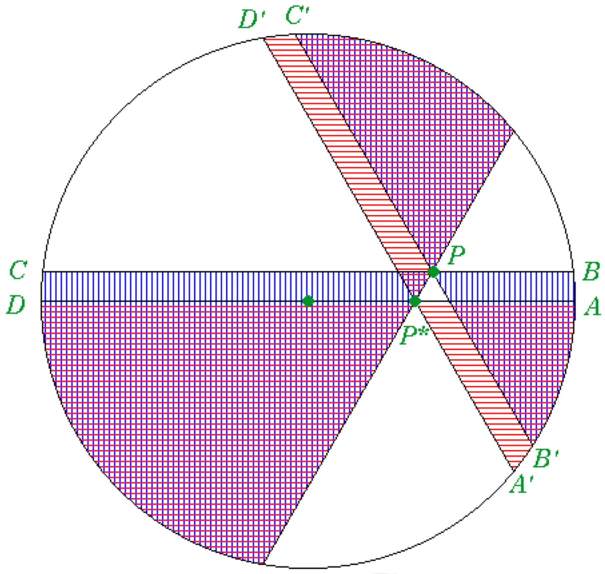

Fig. 3

b.* Begin the same way as above, but restrict P so that $0 \le \theta < \pi/5$, and let P^* denote the intersection of the x-axis with the cut in the direction of $\pi/5$. Form the two pizzas and overlay them, as in figure 4.

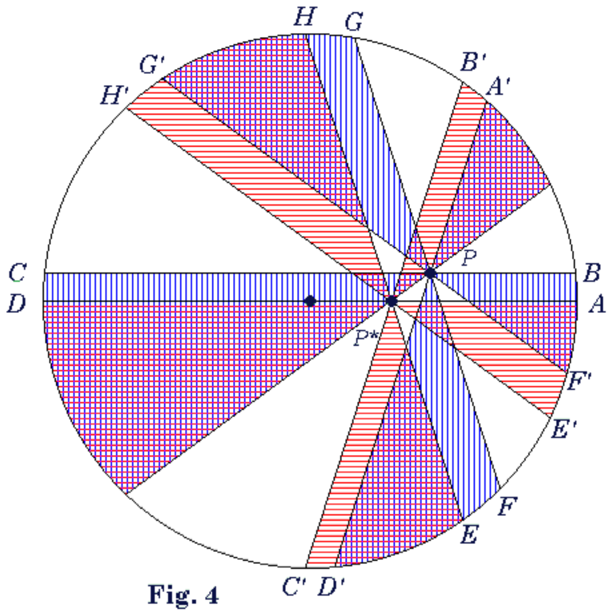

Fig. 4

Notice that the problem now amounts to comparing the sum $ABCD + EFGH$ of the areas of the strips on the first pizza with $A'B'C'D' + E'F'G'H'$ on the second, *plus or minus the holes or overlaps at the intersections* — we need to be careful here, but we will show later that *it suffices to consider the entire strips.* This comparison is no longer trivial, as in part **a.**, for although $ABCD$ and $A'B'C'D'$ have the same "width", as do $EFGH$ and $E'F'G'H'$, the "lengths" of these strips do not coöperate — $ABCD$ is larger than $A'B'C'D'$ while $EFGH$ is smaller than $E'F'G'H'$.

Let us delay the problem of the intersections for now, and show that $ABCD + EFGH \leq A'B'C'D' + E'F'G'H'$. We need to find the area of a strip, by which we mean the difference of two segments of the unit circle. Consider a strip at perpendicular distance D from the center of the circle, and having width W, as in figure 5. Its area is easily found to be $s(D) - s(D+W)$, where $s(x) = \cos^{-1}(x) - x\sqrt{1-x^2}$ is the area of the segment at distance x. (This is the sector OPQ minus the triangle OPQ in figure 6.)

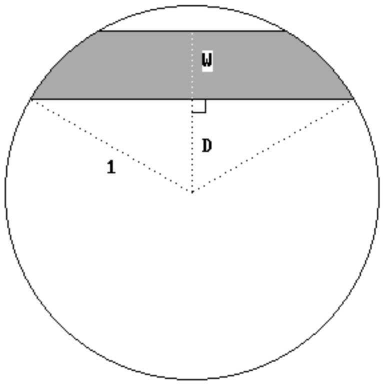

FIGURE 5

Slice of width W
at distance D

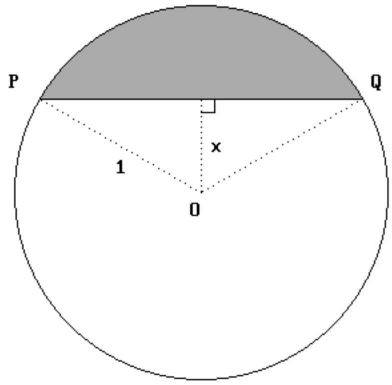

FIGURE 6

Segment at distance x

Returning to figure 4, let w denote the distance from P to P^* and d the distance from P^* to the center [Somehow the version printed in *Math. Mag.* has the d's and w's reversed at this point. We intended for the definitions of d and w to be consistent with the use of $s(D) - s(D+W)$, as mentioned just before the previous figures. So I'm restoring that consistency in this online version], and let $\alpha_j = j\pi/5$, $d_j = d \sin \alpha_j$, $w_j = w \sin \alpha_j$, for $j = 0, 1, 2$. Let

$$\Delta R = \mathcal{R}(|||) - \mathcal{R}(\equiv)$$

and let

$$\Delta S = (ABCD + EFGH) - (A'B'C'D' + E'F'G'H').$$

(We will show later that $\Delta R = \Delta S$.) Then it is easy to see that

$$\begin{aligned} ABCD &= s(d_0) - s(d_0 + w_1), & A'B'C'D' &= s(d_2) - s(d_2 + w_1), \\ EFGH &= s(d_2) - s(d_2 + w_2), & E'F'G'H' &= s(d_1) - s(d_1 + w_2), \end{aligned}$$

and so

$$\Delta S = s(0) + s(d_2 + w_1) + s(d_1 + w_2) - s(d_2 + w_2) - s(w_1) - s(d_1). \quad (1)$$

Next we observe that $s'(x) = -2\sqrt{1 - x^2}$, so we have, with the aid of Taylor's theorem,

$$\begin{aligned} s(x) &= s(0) + \int_0^x s'(t)\, dt \\ &= \frac{\pi}{2} - \int_0^x \left(1 - \frac{t^2}{2} - \frac{t^4}{8} - \frac{t^6}{16} + \cdots\right) dt \\ &= \frac{\pi}{2} - 2x + \frac{x^3}{3} + \frac{x^5}{20} + \frac{x^7}{56} + \cdots, \end{aligned} \quad (2)$$

where all the powers beginning with x^3 are odd and of the same (positive) sign. Applying this to the expression in (1), we see that the contributions of the constant term and the x term cancel in the sum. If each of the remaining terms just happened to yield non-positive contributions, we would be done. This is exactly what happens! Let n be odd and set $g = 2\cos\alpha_1 = 1.681\cdots$ (the golden ratio). Then we have $d_2 = gd_1$ and $w_2 = gw_1$, so that the contribution of x^n in (2) to the expression in (1) is

$$\begin{aligned} & (d_2 + w_1)^n + (d_1 + w_2)^n - (d_2 + w_2)^n - w_1^n - d_1^n \\ =\ & (gd_1 + w_1)^n + (d_1 + gw_1)^n - (gd_1 + gw_1)^n - w_1^n - d_1^n \\ =\ & -w_1^n - d_1^n + \sum_{k=0}^{n} \binom{n}{k} \left[(gd_1)^k w_1^{n-k} + d_1^k (gw_1)^{n-k} - (gd_1)^k (gw_1)^{n-k}\right] \\ =\ & \sum_{k=1}^{n-1} \binom{n}{k} d_1^k w_1^{n-k} \left[g^k + g^{n-k} - g^n\right]. \end{aligned}$$

We're done if

$$g^k + g^{n-k} - g^n \leq 0, \quad \text{for all } n \geq 3 \text{ and } 0 < k < n. \quad (3)$$

In fact, for $n = 3$ we have *equality* for $k = 1, 2$ (since the golden ratio satisfies $g^2 - g - 1 = 0$), and we proceed by induction. Assuming that (3) holds for some $n \geq 3$ and whenever $0 < k < n$, we let $0 < k < n+1$. Then

$$\begin{aligned} g^k + g^{n+1-k} - g^{n+1} &= g(g^{k-1} + g^{n-k} - g^n) \\ &= g(g^k + g^{n-k} - g^n) + g^k - g^{k+1} \\ &\leq g^k - g^{k+1} \qquad \text{(by the induction hypothesis, but only for } 0 < k < n) \\ &\leq 0 \qquad \text{(since } g > 1). \end{aligned}$$

In the case that $k = n$, we get

$$\begin{aligned} g^k + g^{n+1-k} - g^{n+1} &= g^n + g - g^{n+1} \\ &= -g(g^2 - g - 1) - g^2(g-1)(g^{n-2} - 1) \\ &\leq 0, \end{aligned}$$

completing the induction and establishing (3).

[The editors snipped everything in green that follows.] It remains only to show that $\Delta R = \Delta S$, that is, that we can justify ignoring the intersections of the strips. Let us associate the entire strips $ABCD$ and $EFGH$ with $\mathcal{R}(|||)$, despite the fact that they have blank regions and $\mathcal{R}(\equiv)$-regions, and similarly, associate the strips $A'B'C'D'$ and $E'F'G'H'$ with $\mathcal{R}(\equiv)$. Then one may simply verify in each case that the regions in the intersections have the following properties.

(0) Any blank regions in the intersections of the strips belong to the same number of $\mathcal{R}(|||)$-strips as $\mathcal{R}(\equiv)$-strips.

(1) Any $\mathcal{R}(|||)$ regions in the intersections of the strips belongs to one more $\mathcal{R}(|||)$ region than $\mathcal{R}(\equiv)$ regions.

(2) Any $\mathcal{R}(\equiv)$ regions in the intersections of the strips belongs to one more $\mathcal{R}(\equiv)$ region than $\mathcal{R}(|||)$ regions.

(3) Any simultaneously $\mathcal{R}(\equiv)$ and $\mathcal{R}(|||)$ regions in the intersections of the strips belong to the same number of $\mathcal{R}(|||)$-strips as $\mathcal{R}(\equiv)$-strips.

With little trouble, we can see that these properties will allow us to completely fill each strip with its associated region without affecting the difference ΔR — everything balances! Hence, $\Delta R = \Delta S$, the pizza is done and dinner is served.