The measure theoretic approach to the density of quasi-arithmetic sequences

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Most of the preliminaries are found in the paper

"The measure theoretic approach to density," by R. Creighton Buck, *Am. J. Math.* **68** (1946), 560-580.

Def: A *quasi-arithmetic sequence* is a subset of $\mathbb{Z}$ of the form

$$\{\lfloor na + b \rfloor\}_{n \in \mathbb{Z}},$$

where $a > 1$ is irrational and b is real.

Def: The *density* $d(A)$ of a set $A \subset \mathbb{Z}$ is given by

$$d(A) = \lim_{N \to \infty} \frac{\#(A \cap [-N, N])}{2N}$$

when the limit exists.

Note: The results in Buck's paper, and many others deal with measures on the positive integers only, but can easily be extended to all of $\mathbb{Z}$. Conversely, our results can be restated in terms of subsets of $\mathbb{N}$.

Def: Let $\mathcal{D}_0$ be the algebra of subsets of $\mathbb{Z}$ generated by **arithmetic** sequences and finite sets.

Fact: Let m be any translation-invariant measure on $\mathbb{Z}$ such that $m(\mathbb{Z}) = 1$, and containining the class $\mathcal{D}_0$ in its domain of definition. Then

$$m(A) = d(A) \quad \forall A \in \mathcal{D}_0.$$

(The elements of $\mathcal{D}_0$ have the *uniqueness property* relative to the class of all translation-invariant probability measures on $\mathbb{Z}$.)

Note: Such a measure m is only finitely additive. Also, m takes only rational values.

Def: For $S \subset \mathbb{Z}$, let

$$m^*(S) = \inf\{m(A) : S \subset A,\ A \in \mathcal{D}_0\}.$$

Fact: m^* is a translation-invariant outer measure on $\mathbb{Z}$, extending m.

Def: Let $\mathcal{D}_m$ be the class of sets S for which

$$m^*(X) = m^*(X \cap S) + m^*(X \setminus S), \qquad \forall X \subset \mathbb{Z}.$$

Fact: The class $\mathcal{D}_m$ is an algebra of sets; m^* is a translation-invariant finitely additive measure when restricted to $\mathcal{D}_m$.

Def: Let μ be this restriction of m^*, i.e.,

$$\mu : \mathcal{D}_m \to [0, 1],$$

$$\mu(A) = m^*(A), \ \forall A \in \mathcal{D}_m.$$

Fact: The extension is proper, as there are infinite sets of μ-measure zero in $\mathcal{D}_m$. In fact, the range of μ is all of $[0, 1]$.

Fact: $\mu(A) = d(A), \qquad \forall A \in \mathcal{D}_m.$

Fact: The elements of $\mathcal{D}_m$ have the uniqueness property relative to the class of all translation-invariant probability measures on $\mathbb{Z}$.

Fact: There are μ-nonmeasurable subsets of $\mathbb{Z}$, e.g., any $S \subset \mathbb{Z}$ for which

$\#(A \cap S) = \infty = \#(A \backslash S) \quad \forall$ arithmetic series A.

Thus, *all quasi-arithmentic sequences are μ-nonmeasurable*.

Fact: If $S = \{\lfloor na + b \rfloor\}_{n \in \mathbb{Z}}$, with $a > 1$, then $d(S) = 1/a$.

Fact: I am annoyed by the conjunction of the previous two facts.

Question: Given the previous fact, why even mess with μ? Isn't d much better?

Answer: For some things, yes. But the density d is not a measure on the class of sets with density. E.g., $\exists A, B$ each having a density but with $A \cap B$ having undefined density.

Example: Let X be any set with no density.

$X = \{\cdots \bullet\circ\bullet\bullet\circ\circ\bullet\bullet\bullet\bullet\circ\circ\circ\circ\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\circ\circ\circ\circ\circ\circ\circ\circ \cdots\}$

Define A and B by letting

$$A = \{a_n\}_{n\in\mathbb{Z}}, \quad B = \{b_n\}_{n\in\mathbb{Z}},$$

where

$$a_n = 3n, \quad b_n = \begin{cases} 3n & \text{if} \quad n \in X \\ 3n+1 & \text{if} \quad n \notin X \end{cases}$$

Then

$$d(A) = d(B) = 1/3,$$

but

$$A \cap B = 3X,$$

so $A \cap B$ has no density.

Fact: There exist translation-invariant probability measures defined *for all subsets of* $\mathbb{Z}$. (This is a consequence of the Hahn-Banach theorem.)

Def: Let us call such measures "Banach measures" on $\mathbb{Z}$.

Def: We say that $A \subset \mathbb{Z}$ has *"integer-shade equal to* s*"* provided that

$$\nu(A) = d(A) = s$$

for every Banach measure on $\mathbb{Z}$. We then write

$$\mathrm{sh}(A) = s.$$

Proposition: If $a > 1$ and

$$A = \{\lfloor na + b \rfloor\}_{n \in \mathbb{Z}},$$

then

$$\mathrm{sh}(A) = 1/a.$$

(A minor detail of the proof of this assertion is left to a collaborator to be named later.)

Demonstration: Below is some of the set

$$A = A_1 = \{\lfloor n\pi \rfloor\}_{n\in\mathbb{Z}}.$$

$$\begin{aligned} \cdots \quad & , -16, -13, -10, -7, -4, 0, 3, 6, 9, 12, 15, \\ & 18, 21, 25, 28, 31, 34, 37, 40, 43, 47, \\ & 50, 53, 56, 59, 62, 65, 69, 72, 75, 78, 81, \cdots \end{aligned}$$

It is obvious (and even I can prove) that 3 translates of

$$A_1, \quad A_1 + 1, \quad A_1 + 2$$

are disjoint, since the smallest gap is of length 3. Letting ν be any Banach measure on $\mathbb{Z}$, it is clear that

$$\nu(A_1) \leq \frac{1}{3}.$$

Let

$$A_2 = \mathbb{Z} \setminus \biguplus_{0\leq k<3} (A_1 + k).$$

Below is some of A_2.

$$\cdots \quad , -45, -23, -1, 24, 46, 68, 90, 112, 134,$$
$$156, 178, 200, 222, 244, 266, 288, 310,$$
$$332, 357, 379, 401, 423, 445, 467, \cdots$$

This time 22 translates of A_2 are disjoint:

$$(A_2 + j) \cap (A_2 + k) = \emptyset, \quad (0 \leq j < k < 22).$$

(Proof: My calculator told me.) Hence,

$$\nu(A_1) \geq \frac{1}{3}\left(1 - \frac{1}{22}\right) = \frac{7}{22}.$$

Continuing, let

$$B_3 = \mathbb{Z} \setminus \biguplus_{0 \leq k < 22} (A_2 + k).$$

Below is some of B_3.

$$\cdots \quad , -1044, -1043, -1042, -689, -688, -687, \\ -334, -333, -332, 21, 22, 23, 354, 355, 356, \\ 709, 710, 711, 1064, 1065, 1066, \cdots$$

Notice that now there is clumping. The translates of B_3 are not as nice. However, B_3 itself is the disjoint union of three translates of a set A_3, and 333 translates of A_3 are disjoint. It follows that

$$\nu(A_1) \leq \frac{1}{3}\left(1 - \frac{1}{22}\left(1 - \frac{3}{333}\right)\right) = \frac{106}{333}.$$

On the next iteration we let

$$B_4 = \mathbb{Z} \setminus \biguplus_{0 \leq k < 333} (A_3 + k),$$

and find A_4 for which

$$B_4 = \biguplus_{0 \leq j < 22} (A_4 + j),$$

where 355 translates of A_4 are disjoint.

We by now expect to, and do get

$$\nu(A_1) \geq \frac{1}{3}\left(1 - \frac{1}{22}\left(1 - \frac{3}{333}\left(1 - \frac{22}{355}\right)\right)\right) = \frac{113}{355}.$$

Of course we recognize the continued fraction approximations to $1/\pi$,

$$\frac{1}{3}, \frac{7}{22}, \frac{106}{333}, \frac{113}{355}, \ldots$$

Denoting the above sequence by the familiar

$$\left\{\frac{p_n}{q_n}\right\}_{n=1}^{\infty}$$

we "see" that

$$\frac{p_{2n}}{q_{2n}} \leq \nu(A_1) \leq \frac{p_{2n-1}}{q_{2n-1}},$$

and by a well-known theorem, this implies that

$$\nu(A_1) = 1/\pi.$$

Apparently, at the n'th stage, we have to prove (defining $q_0 = q_{-1} = 1$)

$$\nu(B_n) \leq \frac{q_{n-2}}{q_n}.$$

That leads to

$$(-1)^n \, \nu(A_1) \geq (-1)^n \, r_n,$$

where

$$\begin{aligned} r_n &= \frac{1}{q_1}\left(1 - \frac{1}{q_2}\left(1 - \frac{q_1}{q_3}\left(1 - \frac{q_2}{q_4}\left(\cdots\left(1 - \frac{q_{n-2}}{q_n}\right)\cdots\right)\right)\right)\right) \\ &= \frac{1}{q_1} - \frac{1}{q_1 q_2} + \frac{1}{q_2 q_3} - \cdots + (-1)^{n-1}\frac{1}{q_{n-1}q_n} \end{aligned}$$

It is easy to prove that $r_n = p_n/q_n$ by induction. Obviously, $r_1 = 1/q_1 = p_1/q_1$. Supposing the formula holds up to n, we have

$$\begin{aligned} r_{n+1} &= r_n + (-1)^n \frac{1}{q_n q_{n+1}} = \frac{p_n}{q_n} + (-1)^n \frac{1}{q_n q_{n+1}} \\ &= \frac{p_n}{q_n} + \frac{p_{n+1}q_n - p_n q_{n+1}}{q_n q_{n+1}} \\ &= \frac{p_{n+1}}{q_{n+1}} \end{aligned}$$

Similar experiments support the conjecture. The addition of an offset has no apparent effect on the estimates. For example, the same gaps appear in the sequence

$$\{\lfloor n\pi + e \rfloor\}_{n\in\mathbb{Z}},$$

even though the sets A_n are, naturally, different.

Fact? If the numbers $a_1, a_2, \ldots a_m$ are independent, each $a_i > 1$, and if $b_1, b_2, \ldots, b_m$ are real, then

$$d\left(\bigcap_{i=1}^{m} \{\lfloor a_i n + b_i \rfloor\}_{n\in\mathbb{Z}}\right) = \prod_{i=1}^{m} \frac{1}{a_i}.$$

This is probably known. (Reference?)

Problem. Is the above intersection an integer-shading? I have been unable to find experimental evidence of this.

Problem: Does $\mathrm{sh}(A) = s$ imply $d(A) = s$? (I conjecture yes.)

Problem: If the above problem isn't very easy, is it at least true in the case that $s = 0$? In fact, does $d(A) = 0$ imply $\mathrm{sh}(A) = 0$?

Problem: Does every integer-shading contain a nontrivial subshading? Better still, does it contain a subshading of every possible shade? (It suffices to find half-shadings.)

Problem: Is the class of integer shadings an algebra? (I doubt it.)

Problem: Characterise all integer-shadings. That oughta solve the above problems.

Problem: Is it interesting to look at $\mathbb{Z}^k$?