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Subsets of the plane with constant linear shade†

We will construct and examine subsets of the plane which have constant linear shade in each direction, and discuss the possible planar shades of such sets.

For a real number $t \in [0,1]$, we call a set $A \subset \mathbb{R}^2$ a t-shading of the plane (or a set of shade t) if $\mu(A \cap E) = t\ \mu(E)$ whenever $E \subset \mathbb{R}^2$ is Lebesgue measurable and μ is a Banach measure on $\mathbb{R}^2$, i.e., μ is an isometry-invariant extension of the Lebesgue measure defined for all subsets of the plane. We then write $\mathrm{sh}_2(A) = t$. A t-shading A of $\mathbb{R}$ is defined in an analagous fashion, and we write $\mathrm{sh}_1(A) = t$. We may also consider one-dimensional Banach measures defined on an arbitrary line $L \subset \mathbb{R}^2$. If $A \cap L$ is a one-dimensional t-shading, we write $\mathrm{sh}_L(A) = t$. (See [5] for proofs of the existence of t-shadings in $\mathbb{R}$ and $\mathbb{R}^2$.)

As an intuitive aid, we will consider a t-shading of $\mathbb{R}^2$ (resp., $\mathbb{R}$) to be a set for which the probability of hitting the set by a point-dart thrown at $\mathbb{R}^2$ (resp., $\mathbb{R}$) is t.

The following classic example may help motivate the rest of the discussion. In 1920, Sierpiński showed that CH implies that

($\mathcal{S}$) there exists a subset S of the plane with the property that S intersects all horizontal lines in a null set and S intersects all vertical lines in a set whose complement is null.

In terms of throwing darts, we conclude that darts thrown in horizontal directions will almost surely miss S, while darts thrown in vertical directions almost surely strike S. Expressed in other terms, ($\mathcal{S}$) illustrates that there is

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†This presentation has been modified from its original form. It has been edited, sanitized, and for the most part stripped of foolish and irreverent comments, in order to fit this journal.

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no strong Fubini theorem under CH. (Other set-theoretic axioms, independent of ZFC and weaker than CH, also yield $(\mathcal{S})$. See, e.g., [1], [2], [6], [7].)

Lest one be tempted to reject CH in view of the above, we note that $(\mathcal{S})$ is a theorem of ZFC+CH, so AC may be as much to blame[1]. In fact, one may take any well-ordering $\prec$ of $\mathbb{R}$ and let $S = \{(x, y) \in \mathbb{R}^2 : x \prec y\}$. Thus a horizontal section, for any fixed y, has cardinality $< \mathfrak{c}$ while the complements of vertical sections also have this property. If CH holds, these sets are countable, and we certainly then have that $\mathrm{sh}_L(S) = 0$ if L is a horizontal line, while $\mathrm{sh}_L(S) = 1$ if L is vertical. On the other hand it is a fact in ZFC that if $X \subset \mathbb{R}$ and $|X| < \mathfrak{c}$, then $\mathrm{sh}_1(X) = 0$ (see [5, Lemma 4.5]). Thus ZFC is sufficient for such a paradox (in the sense of throwing darts).

The following shows that densities of linear sections of a plane set are even more independent than is suggested by $(\mathcal{S})$.

Proposition 1 *Let $\mathcal{L}$ denote the set of lines in $\mathbb{R}^2$ and let $f : \mathcal{L} \to [0, 1]$ be any function whatsoever. Then there exists $A \subset \mathbb{R}^2$ with the property that for any line $L \subset \mathbb{R}^2$, the set $L \cap A$ is an $f(L)$-shading of L.*

Sketch of proof. Injectively well-order $\mathcal{L}$ as $(L_\beta)_{\beta<\mathfrak{c}}$. For each $\beta < \mathfrak{c}$, let B_β be any $f(L_\beta)$-shading of L_β. If for $\beta < \mathfrak{c}$ the set A_γ has already been constructed for each $\gamma < \beta$, then let $A_\beta = B_\beta \setminus \bigcup_{\gamma<\beta} L_\gamma$. The set $A = \bigcup_{\beta<\mathfrak{c}} A_\beta$ has the desired properties.

In particular, $f(L)$ may have a constant value $t \in (0, 1)$.

Problem 2 *Suppose that A is a subset of the plane whose linear shades are all equal to t. What, if anything, can be said about the planar shade of A? In particular, is it possible for $\mathrm{sh}_2(A) = s \neq t$?*

Being unable to answer the above question, at least we can do this much:

Proposition 3 *For each $t \in [0, 1]$ there exists $T \subset \mathbb{R}^2$ for which*

(1) $\mathrm{sh}_L(T) = t$ *for each* $L \in \mathcal{L}$,

(2) $\mathrm{sh}_2(T) = t$.

Sketch of proof. Let $D = \{(x, y) : x \bmod 1 \in [0, t)\}$. Let f be a non-trivial automorphism of the complex plane (see [3] or [4]). This is a bijective discontinuous function $f : \mathbb{C} \to \mathbb{C}$ satisfying

$$f(z_1 z_2) = f(z_1) f(z_2) \text{ and } f(z_1 + z_2) = f(z_1) + f(z_2), \quad \forall z_1, z_2 \in \mathbb{C}.$$

[1]In the live version, the speaker half-jokingly chided his friends in the real analysis community for not rejecting any set-theoretic axioms which allow $(\mathcal{S})$ and accused them of living in "hip denial" of such paradoxes.

The set $T = f^{-1}(D)$ does the trick. Property **(1)** follows from the fact that for all nonempty open sets U in $\mathbb{C}$, the set $f^{-1}(U) \cap L$ is dense in every line $L \subset \mathbb{C}$. The additive property of f can be used to prove property **(2)**.

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References

[$\mathcal{A}$] BRUCKNER, A. and CEDER, J., On improving Lebesgue measure, *Normat. Nordisk Matematisk Tidskrift.*, **23**(1975), 59-68.

[1] FREILING, C., Axioms of symmetry: throwing darts at the real number line, *J. Symb. Logic*, **51** (1986), no.1, 190-200.

[2] FRIEDMAN, H., A consistent Fubini-Tonelli theorem for nonmeasurable functions, *Illinois J. Math.*, **24** (1980), no.3, 390-395.

[3] KESTELMAN, H., Automorphisms of the field of complex numbers, *Proc. London Math. Soc., II. Ser.*, **53** (1951), 1-12.

[4] KUCZMA, M., *An introduction to the theory of functional equations and inequalities*, Panstwowe Wydawnictwo Naukowe, Warszawa, 1985.

[5] MABRY, R., Sets which are well-distributed and invariant relative to all isometry-invariant total extensions of Lebesgue measure, *Real Anal. Exchange*, **16** (1991-1992), no. 2, 425-459.

[6] SHIPMAN, J., Cardinal conditions for strong Fubini theorems, *Trans. Amer. Math. Soc.*, **321** (1990), no.2, 465-481.

[7] ZAKRZEWSKI, P., Strong Fubini axioms from measure extension axioms, *Comment. Math. Univ. Carolin.*, **33** (1992), no.2, 291-297.

[2]To "wit": Referring to the paper [$\mathcal{A}$], the speaker wondered aloud whether its authors had actually succeeded in improving Lebesgue measure. From the audience, Andy quickly affirmed, "Oh yes, it's much better now!"