This is a re-creation of a talk given at the 37th International Symposium on Functional Equations, May 18, 1999 (during Functional Equations Week in Huntington, West Virginia), at Marshall University. It is slightly edited for a nonverbal presentation, with some irrelevant and impertinent comments deleted, some added.

Abstract: When $f : \mathbb{R} \to \mathbb{R}$ is a continuous solution to the CFE, then f is linear. In this case (if f is not identically 0), whenever $E \subset \mathbb{R}$ is a Lebesgue measurable subset with linear density $d(E) = \alpha$, it is clear also that $d(f(E)) = \alpha$ and $d(f^{-1}(E)) = \alpha$. Here,

$$d(E) = \lim_{\lambda(J)\to\infty} \lambda(E \cap J)/\lambda(J),$$

where λ is Lebesgue measure and J is an interval. It turns out that there are some analogous results when f is discontinuous, with d replaced by a density that is (a shade) different. Further, when $f : \mathbb{C} \to \mathbb{C}$ is a nontrivial Segre function, subsets of the plane with interesting[a] geometric properties are produced as inverse images of some simple subsets of the plane having asymptotic density.

[a]or maybe I should say, "perhaps not unamusing" (you had to be there)

Consider the set in $\mathbb{R}$:

$$B = \mathbb{Z} + [0, 1/3)$$

Clearly, B has asymptotic linear density 1/3:

$$\lim_{m(J)\to\infty} \frac{m(B \cap J)}{m(J)} = \frac{1}{3}.$$

Here B is "one-third" of the real line in the sense that

$$\mathbb{R} = B \uplus (B + 1/3) \uplus (B + 2/3).$$

That is, $\mathbb{R}$ is the disjoint union of three mutually congruent sets. (We use the symbol $\uplus$ to indicate disjoint unions.)

What follows is a proof, using the above idea, that the asymptotic linear density of B is $1/3$. (The proof is trivial in a more tradition way; the following is given as a model for what is to come, which involves nonmeasurable sets.)

Let J be a large bounded interval, $t_1 = 1/3$, $t_2 = 2/3$. (And now forget the actual values of t_1 and t_2, remembering only that $\mathbb{R}$ is the disjoint union of B, $B + t_1$ and $B + t_2$.) Let

$$J^+ = J \cup (J - t_1) \cup (J - t_2),$$
$$J^- = J \cap (J - t_1) \cap (J - t_2),$$

and notice that

$$m(J^+) - T \leq m(J) \leq m(J^-) + T,$$

where $T = \max(t_1, t_2)$ and m is Lebesgue measure.

Then

$$\begin{aligned}
m(J) &= m(J \cap \mathbb{R}) \\
&= m(J \cap (B \uplus (B + t_1) \uplus (B + t_2))) \\
&= m(J \cap B) + m(J \cap (B + t_1)) + m(J \cap (B + t_2)) \\
&= m(J \cap B) + m((J - t_1) \cap B) + m((J - t_2) \cap B) \\
&\leq m(J^+ \cap B) + m(J^+ \cap B) + m(J^+ \cap B) \\
&= 3m(J^+ \cap B) \\
&\leq 3(m(J \cap B) + T)
\end{aligned}$$

Similarly, by considering J^- we also have

$$m(J) \geq 3(m(J \cap B) - T),$$

hence

$$\frac{1}{3} - \frac{T}{m(J)} \leq \frac{m(J \cap B)}{m(J)} \leq \frac{1}{3} + \frac{T}{m(J)},$$

and so

$$\lim_{m(J)\to\infty} \frac{m(J\cap B)}{m(J)} = \frac{1}{3}.$$

> Think of the asymptotic density of B as also being the probability of hitting B with an imperfectly aimed dart, as the distance from the target gets arbitrarily large.

The critical fact: $m(J) \gg T$ by varying J, leaving T fixed. So what if we can vary T, leave J fixed, and still have $T \ll m(J)$? Answer: virtually the same thing.

Example: Let H be any Hamel basis (over $\mathbb{Q}$) for $\mathbb{R}$. For $X \subset \mathbb{R}$, let $E(X)$ denote the "rational span" of X, i.e.,

$$E(X) = \left\{ \sum_{i=1}^{n} q_i x_i : q_i \in \mathbb{Q}, x_i \in X \right\}.$$

Fix $h_0 \in H$.

Let

$$A = E(H \setminus \{h_0\}) + h_0((\mathbb{Z} + [0, 1/3)) \cap \mathbb{Q}).$$

(Note the presence of the earlier set B in the above.) Then for any

$$t_1 \in E(H \setminus \{h_0\}) + (1/3)h_0,$$

$$t_2 \in E(H \setminus \{h_0\}) + (2/3)h_0,$$

we have

$$\mathbb{R} = A \uplus (A + t_1) \uplus (A + t_2).$$

Since the t_1 and t_2 may be selected from dense sets, we have, as before,

$$\frac{1}{3} - \frac{T}{\mu(J)} \le \frac{\mu(J\cap A)}{\mu(J)} \le \frac{1}{3} + \frac{T}{\mu(J)},$$

except now, this holds for arbitrarily small T, so we can write (with no limits)

$$\frac{\mu(J\cap A)}{\mu(J)} = \frac{1}{3}.$$

Perhaps it should be mentioned that μ is not the Lebesgue measure!

The above holds for *any* μ that is a translation-invariant extension of Lebesgue measure to the power set of $\mathbb{R}$.

We call such measures *Banach measures* on $\mathbb{R}$. (These measures are only finitely additive; it is well-known that such measures exist also on $\mathbb{R}^2$, but not on $\mathbb{R}^n$ for $n \geq 3$.)

> The set $A \cap J$ can be thought of as one-third of the interval J, to within an arbitrarily small amount. A itself can be thought of as one-third of the real line itself. I think of it as a "uniformly gray" set with "shade" 1/3. Also, I visualize throwing a dart at the real line. Intuitively, the dart will hit the set A with probability 1/3, no matter how close you aim, if there is any error in your aim whatsoever.

Let's put it this way: now matter how you measure it, so long as you *can* measure it (with translation-invariant extensions of Lebesgue measure),

the shade of A is 1/3.

Def'n: For $s \in [0, 1]$, if

$$\mu(J \cap A) = s\, \mu(J)$$

for all Banach measures μ and all intervals J, we say that A has *shade* equal to s and we write

$$\mathrm{sh}A = s.$$

BTW, it is easy to extend the above construction to obtain sets A for which

$$\mathrm{sh}(A) = s,$$

for any $s \in [0, 1]$. (See [M, 1991].)

In fact one can construct continuously varying shadings of the line: if $g(x)$ is continuous, then $\exists G \subset \mathbb{R}$ for which

$$\mu(J \cap G) = \mu(J) \int_J g(x)dx$$

for all intervals J and all Banach measures μ.

Of course, having $\mu(J \cap A) = s\ \mu(J)$ for all intervals J, it is easy to show that

$$\mu(E \cap A) = s\ \mu(E)$$

for all Lebesgue measurable E.

Also, in $\mathbb{R}^2$, taking $A \times \mathbb{R}$ or $\mathbb{R} \times A$ gives two-dimensional shadings of the plane using Banach measures on $\mathbb{R}^2$.

Now consider the Cauchy Functional Equation (CFE)

$$f(x+y) = f(x) + f(y),$$

where $f : \mathbb{R}^N \to \mathbb{R}$.

Solutions to the above are called *additive functions*. In 1821, A.L. Cauchy proved that (for N=1) the continuous additive functions are all of the form

$$y = mx$$

(despite what your students often suggest).[1]

In 1905, Hamel published the construction of his basis and used it to find a discontinuous solution of the CFE (again N=1). Discontinuous solutions of the CFE are sometimes called *Hamel functions*.

[1] In the talk, I only-two-thirds-jokingly conjectured that the study of the CFE began when a student of Professor Cauchy presented his teacher with an assertion that $\sin(x+y) = \sin(x)+\sin(y)$. This was, perhaps, a regrettable remark on my part, but on the other hand, most of the comments following the talk were related to this comment. Professor Ger mentioned a popular cancellation technique

$$\frac{\sin\ x}{n} = 6$$

(cancel the n's). Maybe sensing a flood of further witnessing, Professor Axcel preemptively pointed out that despite the incessant whining of professors about their students, sometimes the pupil seems to have an innate sense of certain functional equations. I think the example he gave was

$$f(x+y)f(x-y) = (f(x)+f(y))(f(x)-f(y)),$$

which students can verify when $f(x) = \sin(x)$. But pardon the digression. Again.

Some basic CFE facts (see, e.g., Kuczma [2]):

- $f(q\vec{x}) = qf(\vec{x})$ for any additive f and any $q \in \mathbb{Q}$.
- For arbitrary N, continuous additive functions are of the form

$$f(\vec{x}) = \vec{c} \cdot \vec{x},$$

  where $\vec{c}$ is a constant vector in $\mathbb{R}^N$.
- The graph of a Hamel function is dense in $\mathbb{R}^{N+1}$.
- The set $f^{-1}(J)$ is saturated nonmeasurable in $\mathbb{R}^N$ whenever f is a Hamel function and J is a nondegenerate bounded interval in $\mathbb{R}$. In particular, $f^{-1}(J)$ is dense.
- If f is Hamel and not one-to-one, then $f^{-1}(y)$ is dense for each $y \in f(\mathbb{R}^n)$.
- For any Hamel basis H on $\mathbb{R}^N$ and any function

$$g : H \to \mathbb{R},$$

  there exists a unique additive extension f of g to all of $\mathbb{R}^N$. In other words, if f is additive, it is determined by its values on a Hamel basis.

Using this last fact, let H be a Hamel basis for $\mathbb{R}$, fix $h_0 \in H$ and define g on H by

$$g(h) = \begin{cases} 1 & \text{if} \quad h = h_0 \\ 0 & \text{if} \quad h \neq h_0 \end{cases}$$

and extend g to an additive f on $\mathbb{R}$. As before, let

$$B = [0, 1/3) + \mathbb{Z}.$$

Then it is easy to verify that the shading A constructed earlier can be expressed as

$$A = f^{-1}(B).$$

In fact, it is easy to show that for the same f,

$$\mathrm{sh} f^{-1}([0, s) + \mathbb{Z}) = s,$$

for any $s \in [0, 1]$.

Here, the asymptotic linear density d of the set

$$B_s := [0, s) + \mathbb{Z}$$

is equal to s, so we have in this case that

$$\mathrm{sh}(f^{-1}(B_s)) = d(B_s) = s.$$

Obviously, similar additive functions can be constructed by considering different basis elements than h_0, with the same result.

But the result holds for all Hamel f:

Theorem. Let f be any Hamel function. Let $a \in \{2, 3, 4, \ldots\}$, and let

$$A = f^{-1}(\mathbb{Z} + [0, 1/a)).$$

Then

$$\mathrm{sh}(A) = 1/a.$$

I omitted the entire proof during the talk.

Proof. Let J be any nonempty bounded interval. We must show that

$$\mu(J \cap A) = \frac{1}{a}\, \mu(J).$$

Fix any positive integer $N \gg a$, let $x_0 = 0$ and for each $i = 1, 2, \ldots, a-1$, choose

$$x_i \in f^{-1}\left[\frac{-i}{a}, \frac{-i}{a} + \frac{1}{N}\right)$$

such that

$$0 < x_i < \mu(J)/N.$$

Note! For some f, the sets $f^{-1}(-i/a)$ are dense, and in that case the proof is much easier (as in the first example of shade 1/3). But in general, $f^{-1}(-i/a)$ may be empty or be a singleton.

Let

$$J^+ = \bigcup_{i=0}^{a-1} (J + x_i),$$

and observe that

$$\mu(J^+) < \left(1 + \frac{1}{N}\right) \mu(J)$$

and that

$$x_i + f^{-1}\left(\left[\frac{i}{a}, \frac{i+1}{a}\right) + \mathbb{Z}\right) \subseteq f^{-1}\left(\left[0, \frac{1}{a} + \frac{1}{N}\right) + \mathbb{Z}\right).$$

Note: In general,

$$f^{-1}(A) + f^{-1}(B) \subseteq f^{-1}(A + B),$$

but the reverse inclusion need not hold.

Then

$$\begin{aligned}
\mu(J) &= \mu(J \cap \mathbb{R}) = \mu\left(J \cap \biguplus_{i=0}^{a-1} f^{-1}\left(\left[\frac{i}{a}, \frac{i+1}{a}\right) + \mathbb{Z}\right)\right) \\
&= \sum_{i=0}^{a-1} \mu\left(J \cap f^{-1}\left(\left[\frac{i}{a}, \frac{i+1}{a}\right) + \mathbb{Z}\right)\right) \\
&= \sum_{i=0}^{a-1} \mu\left((x_i + J) \cap (x_i + f^{-1}\left(\left[\frac{i}{a}, \frac{i+1}{a}\right) + \mathbb{Z}\right))\right) \\
&\leq \sum_{i=0}^{a-1} \mu\left((x_i + J) \cap f^{-1}\left(\left[0, \frac{1}{a} + \frac{1}{N}\right) + \mathbb{Z}\right)\right) \\
&\leq \sum_{i=0}^{a-1} \mu\left(J^+ \cap f^{-1}\left(\left[0, \frac{1}{a} + \frac{1}{N}\right) + \mathbb{Z}\right)\right) \\
&= a\,\mu\left(J^+ \cap f^{-1}\left(\left[0, \frac{1}{a} + \frac{1}{N}\right) + \mathbb{Z}\right)\right) \\
&= a\,\mu(J^+ \cap A) + a\,\mu\left(J^+ \cap f^{-1}\left(\left[\frac{1}{a}, \frac{1}{a} + \frac{1}{N}\right) + \mathbb{Z}\right)\right).
\end{aligned}$$

To handle the the last term, let $t_1 = 0$ and choose

$$t_i \in f^{-1}\left(\left[\frac{i-1}{N-1}, \frac{i}{N-1} - \frac{1}{N}\right)\right)$$

with

$$0 \leq t_i < \frac{\mu(J)}{N}$$

for $i = 2, 3, \ldots, N-1$. Then

$$f^{-1}\left(\left[\frac{1}{a}, \frac{1}{a} + \frac{1}{N}\right) + \mathbb{Z}\right) + t_i$$
$$\subset f^{-1}\left(\left[\frac{1}{a} + \frac{i-1}{N-1}, \frac{1}{a} + \frac{i}{N-1}\right) + \mathbb{Z}\right)$$

and these last sets are $N-1$ disjoint sets whose union is $\mathbb{R}$. Letting

$$J^{++} = \bigcup_{i=1}^{N-1} (J^+ + t_i)$$

we get

$$\begin{aligned}
& (N-1)\mu(J^+ \cap f^{-1}\left(\left[\frac{1}{a}, \frac{1}{a} + \frac{1}{N}\right) + \mathbb{Z}\right)) \\
= \; & \sum_{i=1}^{N-1} \mu((J^+ + t_i) \cap (f^{-1}\left(\left[\frac{1}{a}, \frac{1}{a} + \frac{1}{N}\right) + \mathbb{Z}\right) + t_i)) \\
\leq \; & \sum_{i=1}^{N-1} \mu((J^+ + t_i) \cap f^{-1}\left(\left[\frac{1}{a} + \frac{i-1}{N-1}, \frac{1}{a} + \frac{i}{N-1}\right) + \mathbb{Z}\right)) \\
\leq \; & \sum_{i=1}^{N-1} \mu(J^{++} \cap f^{-1}\left(\left[\frac{1}{a} + \frac{i-1}{N-1}, \frac{1}{a} + \frac{i}{N-1}\right) + \mathbb{Z}\right)) \\
= \; & \mu(J^{++} \cap \biguplus_{i=1}^{N-1} f^{-1}\left(\left[\frac{1}{a} + \frac{i-1}{N-1}, \frac{1}{a} + \frac{i}{N-1}\right) + \mathbb{Z}\right)) \\
= \; & \mu(J^{++}) < \left(1 + \frac{1}{N}\right) \mu(J^+).
\end{aligned}$$

Putting these together we have

$$\begin{aligned}
\mu(J) \; & < \; a\, \mu(J^+ \cap A) + a \left(\frac{1 + 1/N}{N-1}\right) \mu(J^+) \\
& < \; a \left[\mu(J \cap A) + \frac{1}{N} \mu(J)\right] \\
& \quad + a \left(1 + \frac{1}{N}\right) \left(\frac{1 + 1/N}{N-1}\right) \mu(J).
\end{aligned}$$

Letting $N \to \infty$ gives

$$\mu(J \cap A) \geq \frac{1}{a}\mu(J).$$

Similarly, letting

$$A_k = f^{-1}\left(\left[\frac{k}{a}, \frac{k+1}{a}\right) + \mathbb{Z}\right), \qquad k = 0, 1, 2, \ldots, a-1,$$

we have

$$\mu(J \cap A_k) \geq \frac{1}{a}\mu(J).$$

But

$$\mathbb{R} = \biguplus_{k=0}^{a-1} A_k,$$

so if any of these inequalities are strict, we get

$$\mu(J) = \sum_{k=0}^{a-1} \mu(J \cap A_k) > \sum_{k=0}^{a-1} \frac{1}{a}\, \mu(J) = \mu(J),$$

a contradiction which shows that

$$\mu(J \cap A_k) = \frac{1}{a}\, \mu(J). \qquad \diamond$$

It follows easily that for any Hamel function f and any $M \subset [0, 1)$ which is a disjoint finite union of intervals, that

$$\mathrm{sh} f^{-1}(M + \mathbb{Z}) = m(M). \qquad (\star)$$

The same result holds for any dilation:

$$\mathrm{sh} f^{-1}(r(M + \mathbb{Z})) = m(M) \qquad (r \neq 0)$$

Corollary. If E is a bounded measurable set, then

$$\mathrm{sh} f^{-1}(E) = 0.$$

On the other hand, if $M \subset [0,1)$ is a measurable set, then ($\star$) need not hold for all Hamel f. For example, let $M = [0,1) \cap \mathbb{Q}$ and let f be a Hamel function whose range is $\mathbb{Q}$. Then

$$\begin{aligned} \mathrm{sh} f^{-1}(M + \mathbb{Z}) &= \mathrm{sh} f^{-1}(\mathbb{Q}) \\ &= \mathrm{sh}\mathbb{R} = 1 \neq 0 = m(M). \end{aligned}$$

Theorem. For any $b \in \mathbb{R}$ and any Hamel function f,

$$\mathrm{sh} f^{-1}(b,\infty) = 1/2.$$

Note: For this we must rely on μ which are reflection invariant as well as translation invariant.

The proof was omitted in the talk.

Proof. Let J be a bounded interval and let $\varepsilon > 0$. Choose

$$c \in f^{-1}(-\infty, b) \qquad \text{(dense)}$$

such that c is within ε of the midpoint of J. Define

$$r(x) = 2c - x, \quad J^+ = J \cup r(J)$$

and notice that $\mu(J^+) < \mu(J) + \varepsilon$. Also, $x \in r(f^{-1}(b,\infty))$ iff $x = r(t)$ for some t for which $f(t) > b$. Since $f(c) < b$, we have

$$f(x) = f(2c - t) = 2f(c) - f(t) < 2b - b = b.$$

Thus $x \notin f^{-1}(b,\infty)$, and we have shown that

$$f^{-1}(b,\infty) \cap r(f^{-1}(b,\infty)) = \emptyset.$$

We now have

$$\begin{aligned} 2\mu(J &\cap& f^{-1}(b,\infty)) = \mu(J \cap f^{-1}(b,\infty)) + \mu(r(J) \cap r(f^{-1}(b,\infty))) \\ &\leq& \mu(J^+ \cap f^{-1}(b,\infty)) + \mu(J^+ \cap r(f^{-1}(b,\infty))) \\ &=& \mu(J^+ \cap (f^{-1}(b,\infty) \uplus r(f^{-1}(b,\infty)))) \\ &\leq& \mu(J^+ \cap \mathbb{R}) < \mu(J) + \varepsilon. \end{aligned}$$

It follows that

$$\mu(J \cap f^{-1}(b, \infty)) \leq \frac{1}{2}\mu(J). \qquad \cdots \quad \diamond$$

Fact: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a discontinuous additive function. Then if J is any bounded rectangle in the plane, we again have

$$\mu(J \cap f^{-1}([0, s) + \mathbb{Z})) = s\mu(J),$$

where now μ is any Banach measure (isometry-invariant, defined on all subsets of $\mathbb{R}^2$, extends Lebesgue measure). The entire argument given before goes through virtually unchanged.

Fact: Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a discontinuous additive function. Then there are many other cool facts about f^{-1}(checkerboards, stripes, half-planes, honeycombs...)

Speaking of planar sets...

In 1920, Sierpiński showed that CH implies that there exists a subset S of the plane with the property that S intersects all horizontal lines in a countable set and S intersects all vertical lines in a co-countable set.

Absurd. So CH is false.[2]

Sierpiński's set S is simple to "construct." First, take any well-ordering $\prec$ of $\mathbb{R}$ and let

$$S = \{(x, y) \in \mathbb{R}^2 : x \prec y\}.$$

Thus a horizontal section, for any fixed y, has cardinality $< c$ while the complements of vertical sections also have this property. If CH holds, they are countable.

Let $\mathrm{sh}_1 X$ and $\mathrm{sh}_2 X$ denote the 1- or 2-dimensional shade of a subset X in $\mathbb{R}$ or $\mathbb{R}^2$, repectively.

Fact. For $n = 1$ or $n = 2$, if $X \subset \mathbb{R}^n$ and $|X| < c$, then $\mathrm{sh}_n(X) = 0$.

Simply put, the shade of a noncontinuum set must be zero. This is proved in [3].

Consequence #1: Sierpiński's set S has vertical shades all 1, horizontal shades all 0.

[2] I'm kidding, I'm kidding! (Well, sort of...)

So if you throw darts in any vertical direction, you'll always hit it S. If you throw darts in any horizontal direction, you'll (almost surely) miss S.

Consequence #2: Then AC must also be false.

Consequence #3: Then my research program has just vanished and I must look for work elsewhere.

Consequence #4: Then AC must be okay after all.

That being the case, the following could be useful:

Theorem. Let $\mathcal{L}$ denote the set of lines in $\mathbb{R}^2$ and let

$$f : \mathcal{L} \to [0, 1]$$

be any function whatsoever. Then there exists $A \subset \mathbb{R}^2$ with the property that for any line $L \subset \mathbb{R}^2$, the set $L \cap A$ is an $f(L)$-shading of L.

At this point I described a practical application of such sets. You're in a bar or pub in some part of the world where AC is legal. You offer to play a game with one of the local denizens, wherein you each take turns throwing darts at a set F constructed in the plane of the floor. You each get the same number of throws, first at a set constructed by your opponent, then at one constructed by you. You each sit at right angles to each other, so you throw in the vertical direction, your opponent throws in the horizontal direction. (I didn't provide these details during the talk.) A player who hits F always collects a dollar (or beer, or whatever). The locals invariably construct simple measurable planar sets and the play is fair. But you, with your superior knowledge of AC, of course you construct Sierpiński's set S, so you win on every throw. Two problems emerge. One is that it is impossibly difficult to aim your throws in such exact vertical and horizontal directions. The second is that even if that problem were fixed, you must remember that just because the locals don't know AC doesn't mean that they are stupid. Since they never hit the set at all, they assume you are cheating, so they beat you up, steal your wallet, and throw you out of the joint.

Now here is the practical application of sets such as A in the above theorem. You construct your set so that all lines passing through the dartboard and a vicinity of your chair have a shade of, say, 53/100, while lines through your opponent's vicinity have shade 47/100. This solves the problem of having to aim in an exact direction, as you would for Sierpiński's set S, and also, your opponent doesn't really notice the slight advantage you have. Eventually, you win most of the throws. (You still might get beat up, though.) Here is a crude diagram (dartboard4753.gif) of the idea.

Problem. Suppose that A is a subset of the plane whose linear shades are all equal to t. What is the planar shade of A (if it even exists)?

The "obvious answer" is t. But there is nothing obvious in the AC world, as the following indicates.

Fact: (Kharazishvili (Tbilisi), personal communication) A subset B of the plane exists for which

1) $|B \cap L| < c$ for each line $L \subset \mathbb{R}^2$,

2) $\lambda_2^*(B \cap E) = \lambda_2(E)$ for each Lebesgue measurable set $E \subset \mathbb{R}^2$ (where λ_2^* is the Lebesgue outer measure on $\mathbb{R}^2$),

3) $|B \triangle g(B)| < c$ for all g in the group D_2 of all isometries on $\mathbb{R}^2$.

This implies the existence of a D_2-invariant measure ν (on the σ-algebra generated by B and the Lebesgue measurable sets) which is concentrated on B.

So for any Lebesgue measurable set $E \subset \mathbb{R}^2$,

$$\nu(B \cap E) = \nu(E).$$

This measure can be extended to a Banach measure on $\mathbb{R}^2$.

This means that every linear shade of B is zero, but the planar shade is not.

However, the set B doesn't have a "uniqueness property" in that one can (I think) also define, for the same B, a measure ν' concentrated on $\mathbb{R}^2 \setminus B$.

At least we can do this much:

Theorem: For any $t \in [0, 1]$ there exists $T \subset \mathbb{R}^2$ for which

(1) $\mathrm{sh}_L(T) = t$ for each $L \in \mathcal{L}$,

(2) $\mathrm{sh}_2(T) = t$.

Such a T can be constructed as follows. Let

$$D_t = \{(x, y) : x \in [0, t) \pmod 1\}.$$

Notice that D is just a periodic set of stripes in the plane with asymptotic plane density t. (Any such banded set will do.) Let f be a nontrivial Segre function. This is a discontinuous function $f : \mathbb{C} \to \mathbb{C}$ satisfying

(a) $f(xy) = f(x)f(y)$,

(b) $f(x + y) = f(x) + f(y)$.

(See Kuczma's book [2] or the paper of Kestelman [1].)

The set $T = f^{-1}(D_t)$ does the trick. To prove it, use the fact that for all nonempty open sets U in $\mathbb{R}^2$, the set $f^{-1}(U) \cap L$ is dense in every line L in $\mathbb{C}$. This is used to prove property **(1)**. The additive property **(b)** of f yields the two-dimensional shade (property **(2)**). $\diamond$

It is an open problem whether or not a set $C \subset \mathbb{R}^2$ exists for which all linear shades exist and are equal to some fixed s_1, while the planar shade s_2 also exists but for which $s_1 \neq s_2$ (as opposed to the set in the previous theorem, in which $s_1 = s_2$). It seems inconceivable to me that this can happen, but some experts in measure theory warn me that the one- and two-dimensional measure extensions we are dealing with here are fairly independent.

Added July 2010: The problem above was solved by Jim Roberts (Prof. Emeritus, U. So. Carolina) in March 2008 when he visited LSUS. Hopefully the solution (there exists no such set C) will soon be in print.

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[3] R. Mabry, Sets which are well-distributed and invariant relative to all isometry invariant total extensions of Lebesgue measure, *Real Analysis Exchange*, **16** (1991-1992) 425-459.