PROBLEMS

LOREN C. LARSON, *editor*
St. Olaf College

GEORGE GILBERT, *associate editor*
Texas Christian University

Proposals

To be considered for publication, solutions should be received by March 1, 1993.

1403. *Proposed by Richard Friedlander, University of Missouri–St. Louis, and Stan Wagon, Macalester College, Saint Paul, Minnesota.*

Simpson's aggregation paradox admits a simple baseball interpretation. It is possible for there to be two batters, Veteran and Youngster, and two pitchers, Righty and Lefty, such that Veteran's batting average against Righty is better than Youngster's average against Righty, and Veteran's batting average against Lefty is better than Youngster's average against Lefty, but yet Youngster's combined batting average against the two pitchers is better than Veteran's. Question: Can there be a double Simpson's paradox? That is, is it possible to have the situation just described and, *at the same time*, have it be the case that Righty is a better pitcher than Lefty against either batter, but Lefty is a better pitcher than Righty against both batters combined?

1404. *Proposed by Hillel Gauchman and Ira Rosenholtz, East Illinois University, Charleston, Illinois.*

Find the smallest prime which is not the difference (in some order) of a power of 2 and a power of 3.

1405. *Proposed by Hüseyin Demin, Middle East Technical University, Ankara, Turkey.*

Two circles inscribed in distinct angles of a triangle are *isogonally related* if the tangents from the third vertex not coinciding with the sides are symmetric with respect to the bisector of the third angle. Given three circles inscribed in distinct angles of a triangle, prove that if any two of the three pairs of circles are isogonally related then so is the third pair.

 We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals should be accompanied by solutions, if at all possible, and by any other information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. An asterisk (*) next to a problem number indicates that neither the proposer nor the editors supplied a solution.

Solutions should be written in a style appropriate for Mathematics Magazine. Each solution should begin on a separate sheet containing the solver's name and full address.

Solutions and new proposals should be mailed in duplicate to Loren Larson, Department of Mathematics, St. Olaf College, 1520 St. Olaf Ave., Northfield, MN 55057-1098 or mailed electronically via fax: (507) 663-3549 or e-mail: larson@stolaf.edu.

1406. *Proposed by Detlef Laugwitz, Fachbereich Mathematik Technische Hochschule, Darmstadt, Germany.*

Define a sequence $(a_n)_{n\geq 1}$ by

$$a_1 = \sqrt{3}, \qquad a_n = a_{n+1}(3 - a_{n+1}^2), \qquad 0 < a_n \leq a_1 \qquad \text{for } n = 1,2,3,\dots.$$

Show that $\lim_{n\to\infty} 3^n a_n$ exists and find its value.

1407. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Determine the maximum value of the sum

$$x_1^p + x_2^p + \cdots + x_n^p - x_1^q x_2^r - x_2^q x_3^r - \cdots - x_n^q x_1^r,$$

where p, q, r are given numbers with $p \geq q \geq r > 0$ and $0 \leq x_i \leq 1$ for all i.

Quickies

Answers to the Quickies are on page 272.

Q794. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

The general problem of Appolonius is to draw a circle tangent to three given circles. Special cases ensue when all or some of the circles are replaced by points or lines. Solve the problem in the case of two points O, Q, and a circle C, where O is the center of C and Q is an interior point of C.

Q795. *Proposed by Norman Schaumberger, Douglaston, New York.*

Find a solution in positive integers of

$$a^4 + b^7 + c^9 = d^{11}.$$

Q796. *Proposed by Tadhg Creedon (student) and Desmond MacHale, University College, Cork, Ireland.*

Let $G = \{a_1, a_2, \dots, a_n\}$ be a finite group with binary operation $*$. Define an $n \times n$ matrix $M = (m_{ij})$ by $m_{ij} = 1$ if $a_i * a_j = a_j * a_i$ and $m_{ij} = 0$ otherwise. Show that M is invertible if and only if $n = 1$.

Solutions

Triangular numbers **October 1991**

1378. *Proposed by Stephen G. Penrice, Arizona State University, Tempe, Arizona.*

Let $(i)_j$ denote the falling product $i(i-1)\cdots(i-j+1)$ and let $(i)_0 = 1$. Show that for all positive integers n and k

$$\frac{(n+k)_{k+1}}{2(k!)^2} \sum_{i=1}^{k} (k)_{i-1}(n+k-i)_{k-i}$$

is a triangular number.

I. *Solution by R. S. Tiberio, Natick, Maine.*

Writing $(i)_j$ as $i!/(i-j)!$, we may write the given expression as

$$\frac{1}{2}\binom{n+k}{n}\sum_{i=1}^{k}\binom{n+k-i}{n-1}.$$

Now using the standard additive property of binomial coefficients, we can write this as

$$\frac{1}{2}\binom{n+k}{n}\sum_{i=1}^{k}\left(\binom{n+k-i+1}{n}-\binom{n+k-i}{n}\right),$$

which telescopes to

$$\frac{1}{2}\binom{n+k}{n}\left(\binom{n+k}{n}-1\right),$$

a triangular number.

II. *Solution by Reiner Martin (student), University of California at Los Angeles, Los Angeles, California.*

We claim that

$$n\sum_{i=1}^{k}(k)_{i-1}(n+k-i)_{k-i}=(n+k)_k-k!.$$

The right side gives the number of ways of choosing k elements ordered and without repetition from $\{1,2,\dots,n+k\}$, choosing at least one element from $\{1,2,\dots,n\}$. For such a choice, let the i-th element be the first one chosen from $\{1,2,\dots,n\}$. There are $(k)_{i-1}$ possibilities choosing the first $i-1$ elements from $\{n+1,n+2,\dots,n+k\}$, and n possibilities to choose the ith element from $\{1,2,\dots,n\}$, and $(n+k-i)_{k-i}$ possibilities to choose the last $k-i$ elements from the remaining ones. This proves our identity.

Using $(n+k)_{k+1}=n(n+k)_k$, we now get

$$\frac{(n+k)_{k+1}}{2(k!)^2}\sum_{i=1}^{k}(k)_{i-1}(n+k-i)_{k-i}=\frac{(n+k)_k}{2k!}\left(\frac{(n+k)_k}{k!}-1\right)$$
$$=\frac{1}{2}\binom{n+k}{k}\left(\binom{n+k}{k}-1\right),$$

which is a triangular number.

Also solved by Michael H. Andreoli, Seung-Jin Bang (Korea), Harvey L. Berger, J. C. Binz (Switzerland), William Chen and Edward T. H. Wang, Con Amore Problem Group (Denmark), Robert L. Doucette, Arne Fransén (Sweden), Russell Jay Hendel, Albert Kurz (student), Carl Libis, Norman Lindquist, Jack McCown, James L. Parish, Volkhard Schindler (Germany), Heinz-Jürgen Seiffert (Germany), John S. Sumner, Michael Vowe (Switzerland), Robert J. Wagner, Chris Wildhagen (The Netherlands), David Zhu (student), and the proposer.

Self-seeking operations **October 1991**

1379. *Proposed by John O. Kiltinen, Northern Michigan University, Marquette, Michigan.*

Call an operation $*$ on a nonempty set A *self-seeking* if every permutation of A is an automorphism from $(A, *)$ to $(A, *)$. Such operations have no isomorphic copies on the set other than themselves. Describe all the self-seeking operations, if any, on A.

Solution by Fred Dodd, University of South Alabama, Mobile, Alabama.

The operations $*_1$ and $*_2$ given by $x *_1 y = x$ and $x *_2 y = y$ for each $x, y \in A$ are self-seeking since $\sigma(x *_1 y) = \sigma(x) = \sigma(x) *_1 \sigma(y)$ and $\sigma(x *_2 y) = \sigma(y) = \sigma(x) *_2 \sigma(y)$ for each permutation σ. We show that if $|A| \neq 2, 3$ these exhaust the self-seeking operations; and for $|A| = 3$, $|A| = 2$, there are three and four self-seeking operations respectively. The case $|A| = 1$ is trivial and so we assume that $|A| \geq 2$.

Lemma. If a, b are two distinct elements of A and $*$ is self-seeking then $*$ is completely determined by the values $a * a$ and $a * b$. Moreover, if $|A| > 2$ then $a * a = a$; and if $|A| > 3$ then $a * b = a$ or $a * b = b$.

Proof of Lemma. If x and y are any two distinct elements of A there is a permutation τ with $\tau(a) = x$ and $\tau(b) = y$. Thus, $x * x = \tau(a) * \tau(a) = \tau(a * a)$ and $x * y = \tau(a) * \tau(b) = \tau(a * b)$. If $|A| > 2$ there is a permutation ϕ such that $\phi(a) = a$ and $\phi(a * a) \neq a * a$ when $a * a \neq a$. Thus, since $a * a = \phi(a) * \phi(a) = \phi(a * a)$ it follows that $a * a = a$ when $|A| > 2$. If $|A| > 3$ there is a permutation θ such that $\theta(a) = a$, $\theta(b) = b$; and $\theta(a * b) \neq a * b$ when $a * b \notin \{a, b\}$. Thus, since $a * b = \theta(a) * \theta(b) = \theta(a * b)$ we must have $a * b = a$ or $a * b = b$ when $|A| > 3$.

It is immediate from the lemma that $*_1$ and $*_2$ are the only self-seeking operations when $|A| \neq 2, 3$. If $A = \{a, b, c\}$, the lemma says there are at most three self-seeking operations. It is easy to verify that the operation $*$ given by $x * x = x$, $x * y = z$ where $\{x, y, z\} = \{a, b, c\}$ is a third self-seeking operation. Finally, if $A = \{a, b\}$, the lemma says there are at most four self-seeking operations. It is clear that the two operations given by

$$x *_3 x = y, \qquad x *_3 y = x$$

and

$$x *_4 x = y, \qquad x *_4 y = y$$

where $\{x, y\} = \{a, b\}$ give third and fourth self-seeking operations.

Also solved by S. F. Barger, David Callan, Con Amore Problem Group (Denmark), Robert L. Doucette, Furman University Problem Solving Group, William E. Gould, Colonel Johnson Jr., Emil F. Knapp, Dean Larson, Reiner Martin (student), Jean-Marie Monier (France), Raphael S. Ryger, John S. Sumner, Trinity University Problem Group, C. Wildhagen, and the proposer.

Lower bound for positive root **October 1991**

1380. *Proposed by Thoddi C. T. Kotiah, Utica College of Syracuse University, Utica, New York.*

Let $P(x) = x^n - a_1x^{n-1} - a_2x^{n-2} - \cdots - a_n$, where the a_i are nonnegative real numbers, not all zero. Let $s = \sum_{i=1}^{n} a_i$ and $v = \sum_{i=1}^{n} ia_i$. Prove that a lower bound for the positive real zero of $P(x)$ is $s^{s/v}$.

I. *Solution by the Con Amore Problem Group, Royal Danish School of Educational Studies, Copenhagen, Denmark.*

We have to prove that if $t > 0$, and

$$(1/t)^n - a_1(1/t)^{n-1} - a_2(1/t)^{n-2} - \cdots - a_n = 0,$$

or

$$a_1 t + a_2 t^2 + \cdots + a_n t^n = 1, \tag{1}$$

then $s^{s/v} \le 1/t$, or,

$$s^s t^v \le 1. \tag{2}$$

So assume (1), and for $i = 1, 2, \dots, n$ define

$$\alpha_i = a_i/s. \tag{3}$$

Then

$$\alpha_i \ge 0 \; (i = 1, 2, \dots, n), \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i = 1. \tag{4}$$

Now (2) may be written as

$$(st)^{a_1}(st^2)^{a_2} \cdots (st^n)^{a_n} \le 1,$$

or (taking the sth root on both sides, and using (3)),

$$(st)^{\alpha_1}(st^2)^{\alpha_2} \cdots (st^n)^{\alpha_n} \le 1,$$

or again, as

$$\alpha_1(-\log(st)) + \alpha_2(-\log(st^2)) + \cdots + \alpha_n(-\log(st^n)) \ge 0. \tag{5}$$

Now $-\log x$ is a convex function so by Jensen's inequality and (4), the left side of (5) is at least

$$-\log(\alpha_1 st + \alpha_2 st^2 + \cdots + \alpha_n st^n),$$

or (see (3)),

$$-\log(a_1 t + a_2 t^2 + \cdots + a_n t^n),$$

which is 0 (see (1)), so (5) is true and then so is (2).

II. *Solution by Reiner Martin (student), University of California at Los Angeles, Los Angeles, California.*

By an obvious induction on n, using the derivative, we see that P has exactly one positive real zero, and $P' > 0$ at this point. So it is enough to show that $P(s^{s/v}) \le 0$.

By the weighted arithmetic mean–geometric mean inequality, we have

$$\frac{\sum_{i=1}^{n} a_i s^{(n-i)s/v}}{s} \le \left(\prod_{i=1}^{n} s^{a_i(n-i)s/v} \right)^{1/s} = s^{\sum_{i=1}^{n} a_i(n-i)/v} = s^{ns/v-1}.$$

Thus

$$P(s^{s/v}) = s^{ns/v} - \sum_{i=1}^{n} a_i s^{(n-i)s/v} \ge 0.$$

Also solved by Arnold Adelberg and Eugene A. Herman, Robert A. Agnew, Paul Bracken (Canada), Robert L. Doucette, F. J. Flanigan, G. Ladas, F. C. Rembis, Heinz-Jürgen Seiffert (Germany, two solutions), John S. Sumner, and the proposer.

Ladas cites a paper by David H. Anderson, "Estimation and Computation of the Growth Rate in Leslie's and Lotka's Population Models," *Biometrics* 31, September 1975, pp. 701–718, which contains this, and several other bounds, for the unique positive root of this polynomial.

Diophantine equation **October 1991**

1381. *Proposed by Mihály Bencze, Braşov, Romania.*

Find all integer solutions of the following (n and k are positive integers).

a. $(x+y)^{2n+1} = x^{2n} + y^{2n}$

b*. $(x_1 + x_2 + \cdots + x_k)^{kn+k-1} = x_1^{kn} + x_2^{kn} + \cdots + x_k^{kn}$

I. *Solution to (a) by Keith Neu (student), Louisiana State University in Shreveport, Shreveport, Louisiana.*

We will show that the solutions to (a) are $x = y = 0$, or,

$$x = (b+1)\big((b+1)^{2n} + b^{2n}\big),$$

$$y = -b\big((b+1)^{2n} + b^{2n}\big),$$

where b is any integer.

The binomial theorem implies that x and y must be of opposite signs, unless both are zero. Since the equation is symmetric in x and y, without loss of generality, we may assume $x \geq 0$ and $y \leq 0$. Let $z = -y$, so that the equation becomes $(x-z)^{2n+1} = x^{2n} + z^{2n}$, with $x \geq 0$ and $z \geq 0$. The right side of this equation being nonnegative implies $x \geq z \geq 0$. Clearly $x = z = 0$ is a solution, and if $x = z$, then both must equal 0. So assume that $x > z > 0$.

Let $x = ag$, $z = bg$, where $\gcd(a,b) = 1$. We may assume that $g > 0$, and the above assumptions on x and z imply that $a > b > 0$. Inserting $x = (a/b)z$ in our equation and cancelling a factor of z^{2n} reveals

$$z = \frac{b(a^{2n} + b^{2n})}{(a-b)^{2n+1}}.$$

Since a and b are relatively prime, so are $a - b$ and b, as well as $(a-b)^{2n+1}$ and b. Thus, for z to be an integer, $(a-b)^{2n+1}$ divides $a^{2n} + b^{2n}$. We claim this implies $a - b = 1$. Assume $a - b = 1$. Form

$$a - b = \prod_{m=1}^{q} p_m^{i_m} \quad \text{and} \quad a^{2n} + b^{2n} = \prod_{m=1}^{q} p_m^{r_m} \prod_{m=q+1}^{K} p_m^{r_m},$$

with $r_m \geq i_m(2n+1) \geq 3$ for $m = 1, 2, \ldots q$.

We claim that if p_l is one of the primes in the factorization of $a - b$, then $p_l = 2$. Since $a - b \equiv 0 \pmod{p_l}$ and since $\gcd(a,b) = 1$, we may assume $a \equiv j \pmod{p_l}$, and $b \equiv j \pmod{p_l}$, where $0 < j < p_l$. This implies that $a^m + b^m \equiv j(a^{m-1} + b^{m-1}) \pmod{p_l}$, for $m = 1, 2, \ldots, 2n$. By assumption $a^{2n} + b^{2n} \equiv 0 \pmod{p_l}$. Since $\gcd(j, p_l) = 1$, this implies $a^{2n-1} + b^{2n-1} \equiv 0 \pmod{p_l}$. Continuing in this fashion, it is clear that $2 \equiv 0 \pmod{p_l}$, which implies that p_l divides 2, as claimed.

We have shown that

$$a^{2n} + b^{2n} = 2^R \prod_{m=q+1}^{K} p_m^{r_m},$$

for some $R \geq 3$. In particular, 8 divides $a^{2n} + b^{2n}$. On the other hand, since both a and b must be odd, 2 divides $a^{2n} + b^{2n}$, but 4 does not. This contradiction implies that $a - b = 1$.

Inserting $a = b + 1$ into our expression for z and x and recalling that $y = -z$ yields the solution as claimed.

II. *Solution to (a) by the Con Amore Problem Group, Royal Danish School of Educational Studies, Copenhagen, Denmark.*

Let $u = x + y$. For $u = 0$, the equation has one solution, namely $x = 0$, $y = 0$. For $u \neq 0$, let $s = x/u$ and $t = y/u$ and consequently $s + t = 1$. With $x = su$ and $y = tu = (1 - s)u$ the equation becomes

$$u^{2n+1} = s^{2n}u^{2n} + (1 - s)^{2n}u^{2n},$$

which reduces to

$$u = s^{2n} + (1 - s)^{2n} = s^{2n} + (1 - s)^{2n},$$

and we find

$$x = s\left(s^{2n} + (s - 1)^{2n}\right),$$

$$y = -(s - 1)\left(s^{2n} + (s - 1)^{2n}\right). \qquad (*)$$

If s is an integer, x and y are integers. Conversely, if x and y are integers, $u = x + y$ is also an integer, and $s = x/u$ is a rational number. Let $s = p/q$, where p and q are relatively prime integers, and q is positive. We will show that $q = 1$, so that s is an integer.

We have

$$u = \left(\frac{p}{q}\right)^{2n} + \left(\frac{p}{q} - 1\right)^{2n} = \frac{p^{2n} + (p - q)^{2n}}{q^{2n}},$$

and therefore,

$$uq^{2n} = p^{2n} + \left(p^{2n} - \binom{2n}{1}p^{2n-1}q + \cdots - \binom{2n}{2n-1}pq^{2n-1} + q^{2n}\right)$$

$$= 2p^{2n} + qP(p, q),$$

where $P(p, q)$ is a polynomial in p and q with integer coefficients. Therefore q divides $2p^{2n}$, and since p and q are relatively prime, it follows that q divides 2; and because of $q > 0$ we then have $q = 2$ or $q = 1$.

If $q = 2$ then p is odd, and

$$u = \frac{p^{2n} + (p - 2)^{2n}}{4^n}.$$

One of the two consecutive odd numbers p and $p - 2$ must be congruent to 1 and the other to -1 modulo 4. Both p^{2n} and $(p - 2)^{2n}$ are then congruent to 1, and the numerator $p^{2n} + (p - 2)^{2n}$ is congruent to 2 modulo 4. Therefore the denominator 4^n cannot divide $p^{2n} + (p - 2)^{2n}$, and this is a contradiction, since u is an integer. This shows that $q = 2$ is impossible, and we are left with $q = 1$, so that s is an integer.

Besides $x = 0$, $y = 0$, the only possible solutions to the original equation are therefore those given by $(*)$, where s is an arbitrary integer, and it is straightforward to check that they are indeed solutions.

Also solved by Seung-Jin Bang (Korea), Pierre Barnouin (France), Ty Le (two solutions), Jeff Nelson (student), David Stone, John S. Sumner and Kevin L. Dove, Michael Vowe, and the proposer. No solutions were received to part (b).

Zeros of order n **October 1991**

1382. *Proposed by Michael Golumb, Purdue University, West Lafayette, Indiana.*

Suppose f is a real-valued function of class C^∞ near $x_0 \in \mathbf{R}$, and g is a real-valued function of class C^∞ near $f(x_0)$. Prove that if $g \circ f - e$ (e the identity function) has a zero of order n $(1 \leq n \leq \infty)$ at x_0, then $f \circ g - e$ has a zero of the same order at $f(x_0)$.

Solution by Robert L. Doucette, McNeese State University, Lake Charles, Louisiana.

Let $\alpha = g \circ f - e$ and $\beta = f \circ g - e$. For x near x_0, $\beta(f(x)) = f(x + \alpha(x)) - f(x)$.

First assume that $n = 1$. We have $\alpha(x_0) = 0$, which implies $g(f(x_0)) = x_0$. With these we may show that $\beta(f(x_0)) = 0$ and $\beta'(f(x_0)) = \alpha'(x_0) \neq 0$.

Next assume that $1 < n < \infty$. By induction we can show that, for all finite $k \geq 1$, there are functions $\lambda_0, \dots, \lambda_{k-1}$; $\varphi_0, \dots, \varphi_{k-1}$ of class C^∞ near x_0 such that the k-th derivative of $\beta(f(x))$ equals both

$$\alpha^{(k)}(x) f'(x + \alpha(x)) + f^{(k)}(x + \alpha(x)) - f^{(k)}(x) + \sum_{i=0}^{k-1} \alpha^{(i)}(x) \lambda_i(x) \tag{1}$$

and

$$\beta^{(k)}(f(x))(f'(x))^k + \sum_{i=0}^{k-1} \beta^{(i)}(f(x)) \varphi_i(x). \tag{2}$$

Assuming that $\beta^{(0)}(f(x_0)) = \beta^{(1)}(f(x_0)) = \cdots = \beta^{(k-1)}(f(x_0)) = 0$, $1 \leq k \leq n$, we have, by equating (1) and (2) and evaluating at $x = x_0$,

$$\beta^{(k)}(f(x_0))(f'(x_0))^k = \begin{cases} 0 & 1 \leq k < n, \\ \alpha^{(n)}(x_0) f'(x_0) & k = n. \end{cases}$$

Now $\alpha'(x_0) = 0$ implies that $f'(x_0) \neq 0$. Since $\beta^{(0)}(f(x_0)) = 0$, we have, by induction,

$$\beta^{(i)}(f(x_0)) = 0, \qquad i = 0, 1, \dots, n-1, \quad \text{and} \quad \beta^{(n)}(f(x_0)) \neq 0.$$

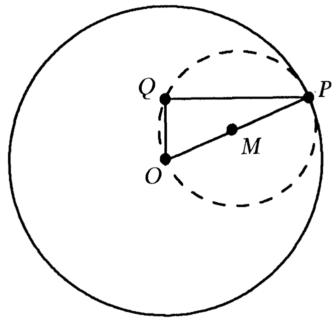
For $n = \infty$ the above argument can be modified to yield $\beta^{(i)}(f(x_0)) = 0$, $i \geq 0$.

Also solved by David Callan, Jean-Marie Monier (France), John S. Sumner, and the proposer.

Answers

Solutions to the Quickies on page 266.

Q794. The center of the desired tangent circle is the midpoint of the hypotenuse of the right triangle drawn as in the figure.



Q795. Since

$$\left(3^{(10!+1)/11}\right)^{11} = 3^{10!+1} = 3^{10!} + 3^{10!} + 3^{10!} = \left(3^{10!/4}\right)^4 + \left(3^{10!/7}\right)^7 + \left(3^{10!/9}\right)^9,$$

we can take $a = 3^{10!/4}$, $b = 3^{10!/7}$, $c = 3^{10!/9}$, and $d = 3^{(10!+1)/11}$.

Q796. The result is trivially true for $n = 1$, so suppose $n > 1$. If there exists $x \in G$, $x \neq x^{-1}$, then, since the centralizer of x^{-1} is equal to the centralizer of x, M has two equal rows, so M is not invertible. So suppose that $x = x^{-1}$ for all $x \in G$. Then, for all $x, y \in G$, $x * y = (x * y)^{-1} = y^{-1} * x^{-1} = y * x$, so G is abelian. But this means that all rows of M are equal, so M is not invertible.