Asymptotic Symmetry of Polynomials

PAUL DEIERMANN
Southeast Missouri State University
Cape Girardeau, MO 63701

RICHARD MABRY
Louisiana State University in Shreveport
Shreveport, LA 71115-2399
rmabry@pilot.lsus.edu

We start with a simple visual observation about polynomials. Consider the polynomial

$$\begin{aligned} P(x) &= x^6 - 140x^5 + 8000x^4 - 238000x^3 + 3870000x^2 - 32400000x + 108000000 \\ &= (x-10)(x-20)^2(x-30)^3 \end{aligned}$$

and some plots of $y = P(x)$ at various scales.

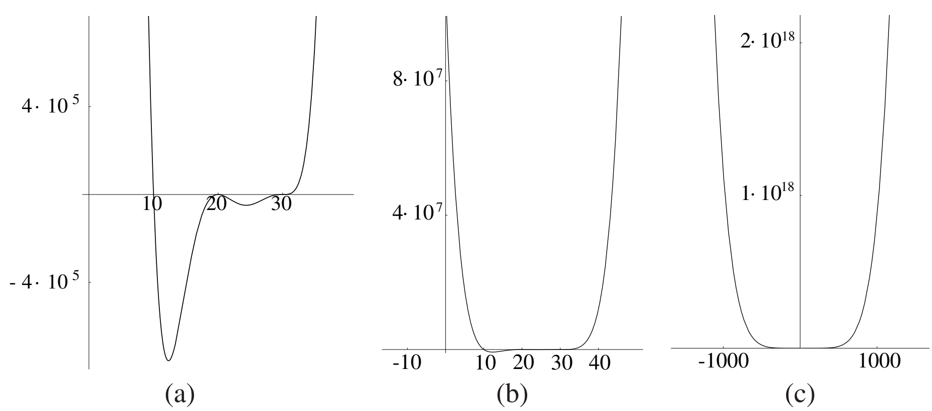

Figure 1 Several views of $P(x)$

In FIGURE 1(b) the graph appears roughly symmetric with respect to $x \approx 25$. In FIGURE 1(c), however, as we zoom out farther, this is no longer evident. In fact, the latter plot resembles that of $y = x^6$, which is the usual observation. What we address here is whether or not the *apparent* line of symmetry in FIGURE 1(b) is genuine, or fails to persist in any meaningful way as we zoom out to scales such as in FIGURE 1(c). Certainly we couldn't detect such an effect visually at such great distances.

A closer look from a distance When the absolute value of x is large, the graph of a polynomial begins to resemble that of the monomial consisting of the polynomial's leading term. Of course, this is justified by the observation that $P(x)$ is asymptotic to x^6. (By this we mean that $\lim_{x\to\pm\infty} P(x)/x^6 = 1$.) For the polynomial $P(x)$ above, this behavior is exhibited by plotting $P(x)$ along with $B(x) = x^6$ in FIGURES 2(a) and 2(b). The graphs of the two functions appear to merge as we zoom out until there is no apparent difference.

Obviously, zooming out far enough would result in the two graphs being indistinguishable, and that would be the case even if we let $B(x)$ be *any* polynomial with the

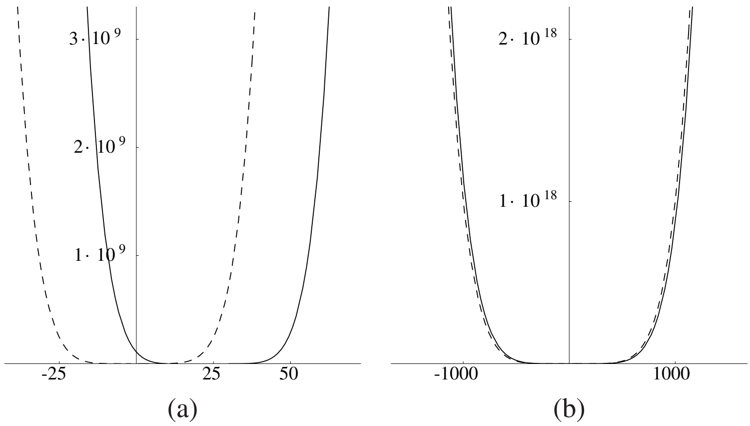

Figure 2 The graphs of P (solid) and B (dotted) from afar

same leading term as $P(x)$. But in the distance chosen in FIGURE 2(a) the graph of P appears decidedly offset, horizontally, relative to B. So the question is, exactly what is this apparent horizontal offset? We might approach it this way: Is there a choice of h so that the graph of $(x-h)^6$ looks *most* like that of $P(x)$? However, since the graph of $(x-h)^6$ is symmetric with respect to the line $x=h$, we are more interested in asking: For what h is the line $x=h$ the axis of asymptotic symmetry of the graph of P? It turns out that for this example, most reasonable interpretations (some presented below) of either of these questions yield the same result, $h = 140/6$. See FIGURE 3.

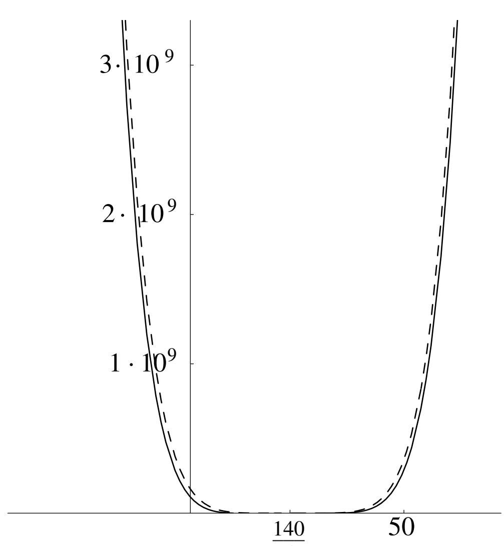

Figure 3 A plot of $P(x)$ (solid) along with the function $C(x) = (x - 140/6)^6$ (dashed)

We shall henceforth assume that $P(x)$ has even degree $n \geq 2$, and, for convenience, that its coefficients are real, with the leading coefficient being unity.

Before we can find the line of asymptotic symmetry, it would be nice if we defined it. One approach would be to choose h so that $P(x)$ and $P(2h-x)$ differ by an amount considered small as $x \to \pm\infty$. For instance, one could require h to satisfy

$$\lim_{x\to\pm\infty}\left[\frac{P(x)-P(2h-x)}{x^{n-1}}\right]=0.$$

Another possibility would be to choose h so $P(x)\approx C(x)=(x-h)^n$. For example, one would require h such that

$$\lim_{x\to\pm\infty}\left[\frac{P(x)-C(x)}{x^{n-1}}\right]=0.$$

(Notice that the usual approximating polynomial $y=x^n$ would not, in general, satisfy the above condition unless the power of x in the denominator was increased by one, and in this case, *any* h would do.)

However, both of these tentative definitions involve a specific power of x in the denominator, hence would not be suitable for generalization to nonpolynomial functions. Our solution is to replace the limit in the domain with a limit in the range, in a way inverting the first potential definition above. We fix the (large) y-coordinate and ask that the corresponding x-coordinates be asymptotically symmetric with respect to some line $x=h$, in the sense described below.

Assume that $\lim_{x\to\pm\infty} f(x)=+\infty$ and that $f(x)$ is eventually monotonic as $x\to\pm\infty$. For k big enough, this will imply that the horizontal line $y=k$ and the graph of f will have exactly two points of intersection, say, $(z_-(k),k)$ and $(z_+(k),k)$. We suppress the dependence upon k in what follows.

DEFINITION. If

$$\lim_{k\to\infty}\frac{z_++z_-}{2}=h,$$

we say $x=h$ is the line of asymptotic symmetry of the graph of f.

In the following theorem we apply this definition to find the line of asymptotic symmetry for polynomials of positive, even degree.

THEOREM. Let $P(x)=\prod_{i=1}^n(x-a_i)$, where the a_is may repeat or may occur as pairs of complex conjugates. Then the line $x=h$ is the line of asymptotic symmetry of P, provided that

$$h=\frac{1}{n}\sum_{i=1}^n a_i.$$

(Notice that h is the average of the roots of the polynomial, weighted by multiplicity.)

Proof. For $|x|$ large enough, P will be a locally one-to-one function whose local inverse has an asymptotic series expansion, valid near $\pm\infty$, being

$$P_\pm^{-1}(y)=\pm y^{1/n}+c_0\pm\frac{c_1}{y^{1/n}}+\frac{c_2}{y^{2/n}}\pm\frac{c_3}{y^{3/n}}+\frac{c_4}{y^{4/n}}\pm\cdots,$$

where the c_is are constants depending on P and $c_0=\frac{1}{n}\sum_{i=1}^n a_i$. (This expansion is justified in the appendix below.) This implies that

$$\begin{aligned}\frac{z_++z_-}{2}&=\frac{P_+^{-1}(k)+P_-^{-1}(k)}{2}\\&=c_0+\frac{c_2}{k^{2/n}}+\frac{c_4}{k^{4/n}}+\cdots,\end{aligned}$$

which approaches $c_0=h$ as $k\to\infty$, and the proof is complete. ∎

It should be noted that the three possible definitions for asymptotic symmetry proffered above might appear different, but (at least) for polynomials, using the asymptotic expression for the $z_\pm$, it is not hard to show all three are equivalent.

Generalizations We conclude this note with some suggested projects for students.

(a) What about asymptotic symmetry with respect to a point? For example, cubic polynomials have symmetry about their inflection points; do all polynomials of odd degree have a point of asymptotic symmetry?

(b) What would be the line (or point) of asymptotic symmetry for rational functions with degree of the numerator larger than the degree of the denominator, or more generally, functions that are written as products of linear factors with real coefficients when the exponents are not necessarily positive integers, so long as the factors are defined for all x and the degree of the resultant algebraic expression is positive? (In this case, the proof given above for polynomials of even degree can be easily modified to handle functions analogous to polynomials of positive, even degree, for instance, $y = x^{1/5}(x-1)^{2/3}/(x-2)^{1/7}$. See FIGURE 4 at the end of this note.) What happens if one allows other types of functions, e.g., $y = \sqrt{x^2 - x^{1/3}}$ or $y = x + \cosh(x)$?

(c) What problems arise if one drops the assumption that the function is eventually monotonic as $x \to \pm\infty$?

Appendix We now justify the expansion in the theorem for the local inverse of $P(x)$. We first find the local inverse for x near positive infinity. For $1 \le k < n$, let b_k be the coefficient of x^k in $P(x)$. So, for example, $b_{n-1} = -nh$, where $h = \frac{1}{n}\sum_{i=1}^{n} a_i$. Then

$$\begin{aligned} y = P(x) &= x^n + b_{n-1}x^{n-1} + \ldots + b_0 \\ &= x^n[1 + L(1/x)], \end{aligned}$$

where $L(1/x) = b_{n-1}/x + \ldots + b_0/x^n$. Let $u = +y^{1/n}$, which is a one-to-one substitution for x near positive infinity, so that $u = x[1 + L(1/x)]^{1/n}$.

Since $L(1/x)$ approaches zero as x approaches infinity, eventually $|L(1/x)| < 1$ for x large enough, so one can expand $[1 + L(1/x)]^{1/n}$ via the binomial theorem. The result is

$$\begin{aligned} u &= x\left[1 + (1/n)L(1/x) + \frac{(1/n)(1/n-1)}{2!}L(1/x)^2 + \cdots\right] \\ &= x[1 + (b_{n-1}/n)(1/x) + (\text{higher powers of } 1/x)] \\ &= x + b_{n-1}/n + (\text{higher powers of } 1/x). \end{aligned}$$

One now solves for the inverse function of the form

$$x = G(u) = u + c_0 + \frac{c_1}{u} + \cdots$$

by forming the equation $u = P(G(u))$ and recursively solving for the coefficients c_0, c_1, etc. In particular, $c_0 = -b_{n-1}/n = h$. The local expansion of the inverse is then given by $x = G\left(y^{1/n}\right)$.

For the local inverse when x is near minus infinity, the only change is that now u becomes $-y^{1/n}$, and the rest is identical.

Encore We cannot resist one more picture. Let

$$f(x) = x^{1/5}(x-1)^{2/3}/(x-2)^{1/7},$$

let $n = 76/105$, and let $h = 10/19$. Below are the graphs of $y = f(x)$ (solid) and $y = (x-h)^n$ (dashed).

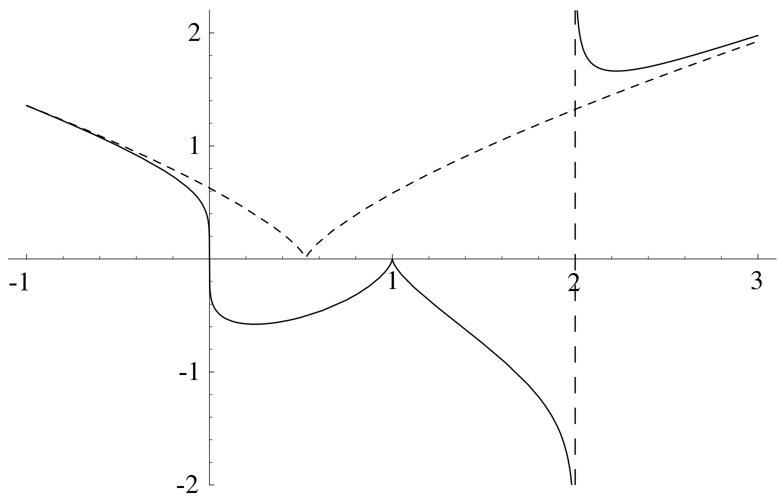

Figure 4 A more general example

Duality and Symmetry in the Hypergeometric Distribution

JAMES JANTOSCIAK
Brooklyn College (CUNY)
Brooklyn, NY 11210
jamesj@brooklyn.cuny.edu

WILLIAM BARNIER
Sonoma State University
Rohnert Park, CA 94928
bill.barnier@sonoma.edu

A CNN posting on the internet [**1**] reports that the 1996 trial of a rap star on manslaughter charges resulted in a hung jury. The posting says that the jury was composed of 7 men and 5 women and was hung at 9 to 3. Not reported is how many men or women voted with the majority. Several interesting probability problems come to mind. For example, what is the probability that exactly 5 jurors among the majority are men?

The usual solution for such a problem utilizes methods that are associated with the hypergeometric probability distribution and involves designating successes and choosing a sample from a population. For the problem stated, it is clear that the population is the 12 jurors. But for the men and the majority, it is not so clear which should be