Problems and Solutions

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Let $S = \{(x, y) : x, y \text{ are rational}, x < y, x^x = y^y\}$. Note that $(1/4, 1/2)$ is in $S$. Find an accumulation point of $S$ or prove that $S$ has no accumulation points.

**704. Proposed by Roger B. Nelsen, Lewis & Clark College, Portland, OR**

We say that a continuous random variable is symmetric about zero if the density function of the random variable is an even function. Let $X$ and $Y$ be identically distributed continuous random variables. Prove or disprove:

(a) The difference $X - Y$ is symmetric about zero.

(b) If $X$ and $Y$ are symmetric about zero, then so is the sum $X + Y$.

Do your answers in (a) or (b) change if $X$ and $Y$ are also assumed to be independent?

**705. Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington College, Abington, PA**

If $0 < a < b$, prove that

$$
\frac{ab}{a^2 + b^2} \leq \frac{2ab}{a + b} \leq \sqrt{ab} \leq \frac{a^2 + b^2}{a + b} \leq b.
$$

**SOLUTIONS**

**A Viewing Window for Limaçons**

**676. Proposed by Rick Mabry, LSU–Shreveport and Paul Deiermann, Lindenwood University**

Let $r(\theta) = 1 + b \cos(\theta)$, where $0 < b \leq 1$, describe a limaçon in polar coordinates. Determine the smallest rectangle of the form $[x_1, x_2] \times [y_1, y_2]$ that contains all these graphs. (This rectangle could be used as a fixed viewing window that contains the graphs of each of the limaçons.)

**Solution by Jack V. Wales, Jr., The Thacher School, Ojai, CA**

We seek the supremum and infimum of $x = r \cos(\theta) = \cos(\theta) + b \cos^2(\theta)$ and $y = r \sin(\theta) = \sin(\theta) + \frac{b}{2} \sin(2\theta)$ over $0 < b \leq 1$, $0 \leq \theta \leq 2\pi$. Since both $x$ and $y$ are continuous functions of $b$ at $b = 0$ we can extend the domain to include $b = 0$.

Since $\cos^2(\theta) \geq 0$, it is clear that $-1 \leq \cos(\theta) \leq \cos(\theta) + \cos^2(\theta) \leq 2$. For $b = 1$ and $\theta = 0$, $x = 2$ and for $b = 0$ and $\theta = \pi$, $x = -1$. Thus $[x_1, x_2] = [-1, 2]$.

Since the graph of the limaçon is symmetric about the $y$ axis, $y_1 = -y_2$. For $0 \leq \theta \leq \frac{\pi}{2}$, $\sin(2\theta)$ is positive and for $\frac{\pi}{2} \leq \theta \leq \pi$, $\sin(2\theta)$ is negative. Thus for each $\theta$ in the latter interval, $y$ attains a maximum when $b = 0$, and for each $\theta$ in the former interval, $y$ attains a maximum when $b = 1$. On $\frac{\pi}{2} \leq \theta \leq \pi$ with $b = 0$, $y$ attains a maximum of $1$ at $\theta = \frac{\pi}{2}$. Thus the maximum value of $y$ will occur on $0 \leq \theta \leq \frac{\pi}{2}$ with $b = 1$. Standard calculus techniques reveal that this happens at $\theta = \frac{\pi}{4}$. Therefore, $[y_1, y_2] = \left[-\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{4}\right]$.

*Also solved by MICHEL BATAILLE, Rouen, France; BEN B. BOWEN, Vallejo, CA; JEREMY CASE, Taylor U.; ROBERT D. CRUCE, Jr.; RICHARD DAQUILA, Muskingum C.; JAMES DEUMMEL, Bellingham, WA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, CT; GREG DRESDEN, Washington and Lee U.; BILL DUNN, III, Montgomery C.; BILL GERSON, Prince Georges C. C.; JOHN GRAHAM, Penn State Wilkes-Barre; RICKY IKEDA, Leeward C. C.; PETER M. JARVIS, Georgia C. & State U.; KIM McINTURFF, Santa Barbara, CA; THOMAS*
An Inverse Function

677. Proposed by Geoffrey A. Kandall, Hamden, CT

The function \( f : (0, \infty) \rightarrow (-\infty, \infty) \) defined by \( f(t) = \frac{\sinh^2(t)}{2\sinh(t)} - \coth(t) \) is increasing and onto. Derive an explicit formula, that involves only algebraic functions and natural logarithms, for the inverse function \( f^{-1} \).

Solution by M. Reza Akhlaghi, Prestonsburg Community College, Prestonsburg, KY

The function \( f \) satisfies

\[
y = f(t) = \frac{(1 + e^{2t})(e^{2t} - 2e^t - 1)}{2e^t(e^{2t} - 1)}
\]

with \( f(\ln(1 + \sqrt{2})) = 0 \). Let \( u = e^t \). Solving for \( u \) in terms of \( y \), we are led to

\[
u^4 - 2(y + 1)u^3 + 2(y - 1)u - 1 = 0.
\]

This equation factors:

\[
\left(u^2 - (y + 1)u - y - \sqrt{y^2 + 1}(u + 1)\right)\left(u^2 - (y + 1)u - y + \sqrt{y^2 + 1}(u + 1)\right) = 0.
\]

The fact that \( y = 0 \) when \( u = 1 + \sqrt{2} \) shows that only the left factor will yield a solution; using the quadratic formula it also shows that

\[
u = \frac{1}{2} \left(y + 1 + \sqrt{y^2 + 1} + \sqrt{(y + 1 + \sqrt{y^2 + 1})^2 + 4\left(y + \sqrt{y^2 + 1}\right)}\right)
\]

is the only acceptable solution. The desired function is \( t = f^{-1}(y) = \ln(u) \).

Also solved by MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian C.; JOSEPH COSTER, Macomb, IL; DANIELE DONINI, Bertinoro, Italy; JAMES DUEMMEL, Bellingham, WA; BILL DUNN, III, Montgomery C.; FLORIDA GULF COAST PROBLEM GROUP, Florida Gulf Coast U; JOHN GRAHAM, Penn State Wilkes-Barre; MURRAY S. KLAMKIN, U. of Alberta; HARRIS KWONG, SUNY C. at Fredonia; KIM MCINTURFF, Santa Barbara, CA; STEPHEN NOLTIE, Ohio U.- Lancaster; WILLIAM SEAMAN, Albright C.; CORNELIUS STALLMAN and GERALD THOMPSON, Augusta State U.; SAMUEL A. TRUITT, JR., Middle Tennessee State U.; LI ZHOU, Polk C.C.; and the proposer.

A Double Sum

678. Proposed by David Atkinson, Olivet Nazarene University, Kankakee, IL

For \( n = 0, 1, \ldots \), find the value of the double sum \( \sum_{j=0}^{n} \sum_{j=0}^{n-1} \frac{(-1)^j}{j^2j^2} \) as a function of \( n \).