Supplement to Fibonacci Numbers, Integer Compositions, and Nets of Antiprisms

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INTRODUCTION. This supplement contains various extensions of discussions of topics found in the article, "Fibonacci Numbers, Integer Compositions, and Nets of Antiprisms" [**3**].

Labels herein prefixed with "M" refer to corresponding labels in that main article. The contents are as follows, with the section numbers from [**3**] added where warranted:

1. Special and degenerate cases. Subsection M2.4. The 3-antiprism (as distinct from the octahedron); the 2-antiprism (as distinct from the tetrahedron) and 1-antiprism (as distinctly absurd).

2. Alternative proof of Lemma M1. Section M3. The coolest part—an induction and recursive generation of the symmetric $(n+1)$-nets from the symmetric n-nets.

3. Path nets. Subsection M5.2. Counting the nets whose spanning tree is a path.

4. Head counts. Some fine structure among the different families.

5. Labeled nets. Kirchhoff's theorem, counting the labeled nets, and a determinant problem.

Warning: Due to the graphic nature of our program (by means of various dismemberments and graftings), reader discretion is advised.

1. SPECIAL AND DEGENERATE CASES. The n-antiprisms when $n = 3$ require special consideration for reasons described below. Consideration of two more "degenerate" cases, $n = 2$ and $n = 1$, aren't necessarily interesting from a geometric point of view, but they are included here because they so nicely fit into our scheme of things in the purely combinatorial sense (they make sense as arrow-grams, at least) and because they can serve as more convenient "base cases" for the inductive proof in the next section, due to the smaller number of objects that must be individually checked.

1.1. Special case: $n = 3$. When $n = 3$ the two 3-gonal heads are congruent to the band's triangular faces, but we consider the heads to be distinct. (If they are not considered distinct, then what we really have are nets of the *regular octahedron*, which has only 11 distinct nets, as does its polyhedral dual, the cube; see [**4**].) Figure 1 shows the complete list of these distinct 3-antiprism nets, with the heads marked in green and various body parts differently shaded. (Necks are red, collars are blue.) Note that, as promised, there are exactly $8 = F_6 = s_3$ symmetric nets among the $36 = s_3(s_3+1)/2 = t_3$ members of the collection $\mathcal{T}_3$.

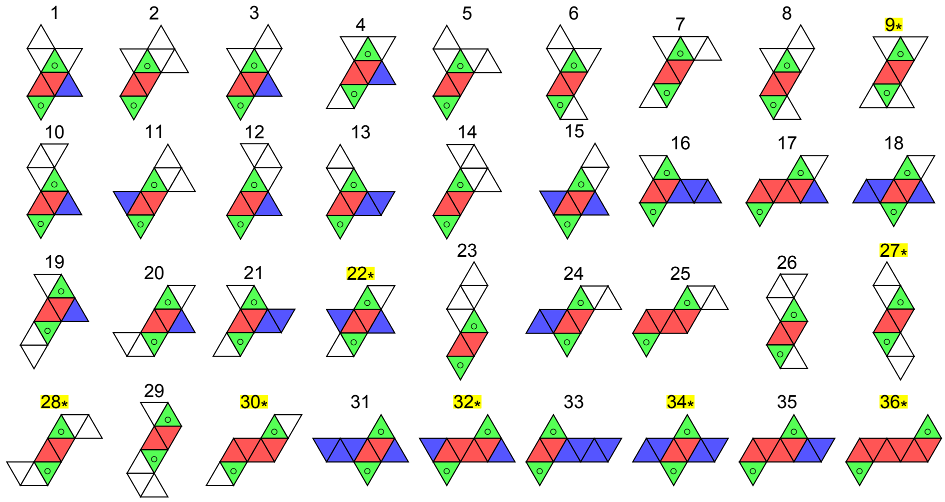


Figure 1. All 36 members of $\mathcal{T}_3$. The 8 members of $\mathcal{S}_3$ are marked (#9, 22, 27, 28, 30, 32, 34, 36).

1.2. Degenerate case: $n = 2$. When $n = 2$, if we want to consider 2-gons (sometimes called *digons*) as viable heads of antiprisms, we obtain nets that are indistinguishable due to the degeneracy of the heads, which become line segments. As plane figures, these are identical to the nets of the *regular tetrahedron*, of which there are exactly 2. As antiprism nets, there are instead exactly $3 = F_4 = s_2$ symmetric ones among the $6 = s_2(s_2+1)/2 = t_2$ members of the collection $\mathcal{T}_2$; see Figure 2. These can be folded into a 2-antiprism (identical to the tetrahedron) with the two "opposite" edges (marked in green) becoming the 2-gonal heads (numbered 0 and 5) of the 2-antiprism.

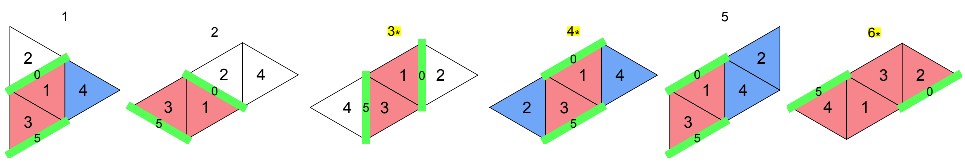


Figure 2. All 6 members of $\mathcal{T}_2$. The 3 members of $\mathcal{S}_2$ are marked (#3, 4, 6).

Arrow-grams for two of the six 2-antiprisms are shown in Figure 3.

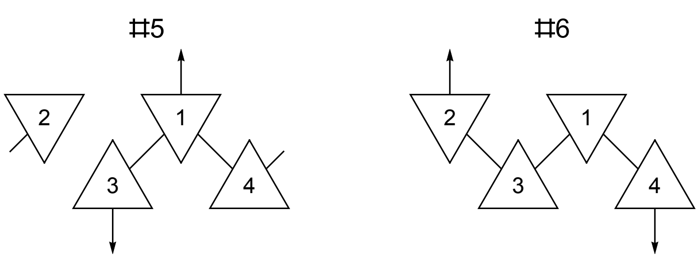


Figure 3. Arrow-grams (left) of nets (right) #5 and #6 of Figure 2.

1.3. Very special case: $n = 1$. We can even insist that when $n = 1$ we have a meaningful situation, though for a plane net or three-dimensional interpretation the heads of a 1-antiprism would need to be considered as 1-gons (points). We leave such head scratching to the reader's imagination. In a strictly combinatorial sense there is no difficulty, and we obtain exactly one 1-antiprism, which is symmetric. Thus $1 = F_2 = s_1$ and $1 = s_1(s_1 + 1)/2 = t_1$, so all's well with our alleged formulas in this case, too.

In Figures 4 and 5 the arrow-grams for the complete 1-antiprism and its only net are shown. Observe also that in Figure 4 we have a multigraph, as there are two edges (the central one and the wrap-around) joining the only two band members (imaginary triangular faces) 1 and 2. Deleting either of these edges gives the same labeled graph, our standardized version of which is shown in Figure 5.

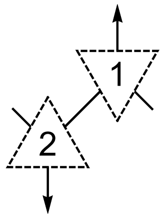


Figure 4. The arrow-gram of the 1-antiprism. Two edges join faces 1 and 2.

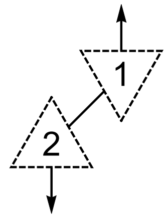


Figure 5. The arrow-gram of the only net of the 1-antiprism.

2. ALTERNATIVE (ORIGINAL) PROOF OF LEMMA M1. This was the original proof found for Lemma M1. (It is what actually inspired the article.) It is elementary and illustrates interesting recursive properties of the objects in question, whether these are the symmetric nets or their surrogates—the compositions of integers used in the first proof. It actually provides a proof of the result (M6), though we do not fashion it that way. And it is a fitting match for the proof of the total count (Theorem M2) in the article [**3**], using similar sensibilities: we'll mutilate and rearrange existing nets (occasionally using spare body parts) to make new ones. Not only that, but with this proof the results are self-contained and the Fibonacci properties are revealed.

We repeat the statement of the lemma below.

Lemma (M1). *For each $n \geq 1$, the number $s_n(h, k)$ of symmetric nets of the n-antiprism having neck size h and lapel size k is given by*

$$s_n(h,k) = \begin{cases} 1 & \textit{if} \quad h + k = n \\ F_{2(n-h-k)} & \textit{if} \quad 1 \leq h + k < n. \end{cases} \tag{M4}$$

Alternative proof of Lemma M1. This proof of the lemma is by induction. The claim of the lemma is embodied in the table of Figure 6, which we refer to as the table of order n. The motivation for the inductive step of our proof can be seen by comparing that table with the one of order $n + 1$ in Figure 7.

$s_n(h,k)$	$h=1$	2	3	$\cdots$	$n-1$	n
$k=0$	F_{2n-2}	F_{2n-4}	F_{2n-6}	$\cdots$	F_2	1
1	F_{2n-4}	F_{2n-6}	$\cdots$	F_2	1	
2	F_{2n-6}	$\cdots$	F_2	1		
$\vdots$	$\vdots$	$\vdots$	$\iddots$			
$n-2$	F_2	1				
$n-1$	1					

Figure 6. The claimed table of values of $s_n(h,k)$.

$s_{n+1}(h,k)$	$h=1$	2	3	4	$\cdots$	n	$n+1$
$k=0$	F_{2n}	F_{2n-2}	F_{2n-4}	F_{2n-6}	$\cdots$	F_2	1
1	F_{2n-2}	F_{2n-4}	F_{2n-6}	$\cdots$	F_2	1	
2	F_{2n-4}	F_{2n-6}	$\cdots$	F_2	1		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$			
$\vdots$	$\vdots$	$\vdots$	$\iddots$				
$n-1$	F_2	1					
n	1						

Figure 7. The table of values of $s_{n+1}(h,k)$ to be established by induction.

In each table let's refer to certain subsets of indices as in Figure 8:

Part A: the columns $h > 1$.
Part B: the entries $k > 0$ in the column $h = 1$.
Part C: the single entry for $h = 1$, $k = 0$.

	1	2 $\cdots$
0	C	
1		A
2	B	
$\vdots$		

Figure 8. The "ABC" layout.

We further let A, B, and C refer to various sets of indices, nets, or just parts of the tables when convenient. Subscripts on those same labels will indicate the order of the table, for instance, B_{n+1} refers to part B of the table of order $n+1$. We'll stretch this notation further by letting $\mathcal{S}(A_n) = \bigcup\{\mathcal{S}_n(h,k) : h > 1\}$, and so on.

Now towards our proof, observe that the values in A_{n+1} (in Figure 7) are identical to those in the combined parts A_n, B_n, and C_n, i.e., the entire table of order n in Figure 6. This suggests we look for one-to-one mappings between the subsets of $\mathcal{S}(A_{n+1})$ and $\mathcal{S}_n = \mathcal{S}(A_n) \cup \mathcal{S}(B_n) \cup \mathcal{S}(C_n)$ that correspond to matching entries of the tables. Similarly, B_{n+1} is identical to the combined parts B_n and C_n, so we'll look for such correspondences between $\mathcal{S}(B_{n+1})$ and $\mathcal{S}(B_n) \cup \mathcal{S}(C_n)$. For the remaining lone cell of C_{n+1}, we have already shown that the sum of all the entries in A_n, B_n, and C_n is equal to the single entry F_{2n} of C_{n+1}. That suggests we find correspondences between $\mathcal{S}(C_{n+1})$ and all of $\mathcal{S}_n$.

For the base case of the induction, all the nets for the cases $n = 1$, 2, or 3 (reader's choice) are accounted for in Figures 5, 2, and 1, respectively, making verification

$s_2(h,k)$	$h=1$	2
$k=0$	$F_2=1$	1
1	1	

$s_3(h,k)$	$h=1$	2	3
$k=0$	$F_4=3$	$F_2=1$	1
1	$F_2=1$	1	
2	1		

Figure 9. Values of $s_n(h,k)$ for $n=2$ and $n=3$.

straightforward. For convenience, Figure 9 shows versions of the table given in Figure 6 for the cases $n=2$ and 3. (The table for $n=1$ consists of a single cell that says $s_1(1,0)=1$.)

To fully check the base case, the numbers of nets of different neck and lapel sizes need to be confirmed, and it is easy to scan Figure 2 or 1 and collect the various (h,k) for this purpose. As an example, the nets witnessing $s_3(1,0)=F_4=3$ are those numbered #9, 27, and 28 in Figure 1. Beyond that, the reader is left on the hook for checking that all symmetric nets for $n=2$ and $n=3$ are indeed given in Figures 2 and 1, the first of these, for $n=2$, being (arguably) easier. (And why not start with $n=1$? If you wish, dear reader, but the pictures probably won't make as much sense.)

We proceed then to the inductive step of the induction and assume that our claim (the table in Figure 6) holds for symmetric antiprisms of order n.

Part A. To each member of $\mathcal{S}_n$ we do the following: without changing its lapel size, we increase its neck size and head size by 1. See the examples in Figures 10 and 11. Compare the arrow-grams and notice that the arrow-gram criteria guarantee that the insertion of a pair of neck vertices into a valid arrow-gram of order n results in a valid arrow-gram of order $n+1$, and therefore gives a net of the antiprism of order $n+1$. Symmetry is preserved and the lapel counts are unchanged. On the resulting net, the heads have an added edge to accomodate this neck-stretching; as the edge is inserted, heads will roll from the center, all the head decorations carried forth by one edge. The lapels remain in place, unruffled by the operation.

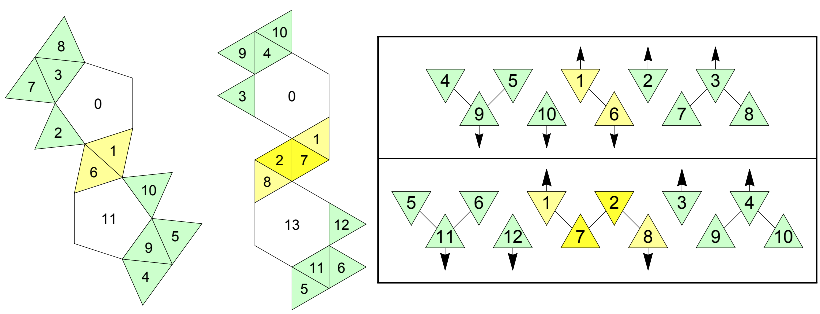


Figure 10. Part A. Increasing the neck size. (Here h goes from 1 to 2; $k=0$.)

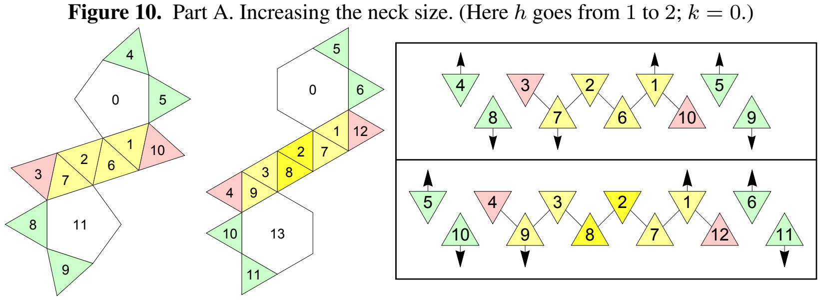


Figure 11. Part A. Increasing the neck size. (Here h goes from 2 to 3; $k=1$.)

We have just described a bijection $f_A : \mathcal{S}_n \leftrightarrow \mathcal{S}(A_{n+1})$ that takes each $\mathcal{S}_n(h,k)$ bijectively to $\mathcal{S}_{n+1}(h+1,k)$, the inverse mapping being clear, as any member of $\mathcal{S}_{n+1}$ having $h > 1$ can have its neck size decremented in the obvious, reversed way. As a consequence, $s_{n+1}(h+1,k) = s_n(h,k)$ for all (h,k) of the nth order table.

Part B. To each member of $\mathcal{S}_n(1,*)$ we increase the lapel size by one without changing the neck size, while increasing the head size by 1. See Figures 12 and 13. The arrow-gram criteria verify that the construction is correct. As a lapel face is added, the head decorations shift one place, accordingly, as a new head edge is inserted. This defines a bijection $f_B : \mathcal{S}(C_n) \cup \mathcal{S}(B_n) \leftrightarrow \mathcal{S}(B_{n+1})$ that bijectively takes each $\mathcal{S}_n(1,k)$ to $\mathcal{S}_{n+1}(1,k+1)$, showing that $s_{n+1}(1,k+1) = s_n(1,k)$ for $0 \le k \le n-1$. We have by now copied parts B and C (the first column) of the nth order table in Figure 6 onto part B of the table of order $n+1$ in Figure 7.

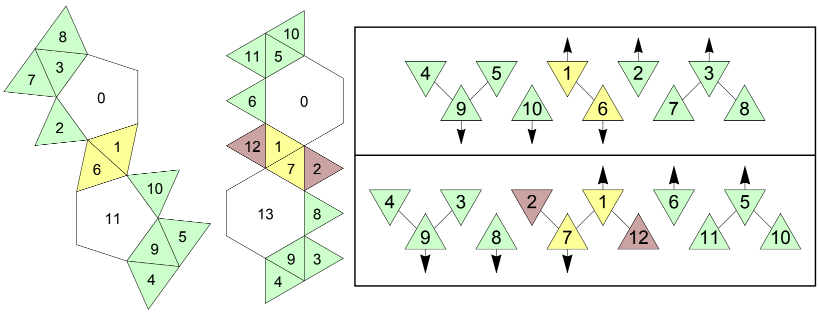


Figure 12. Part B. Increasing the lapel size when $h = 1$. Here k goes from 0 to 1.

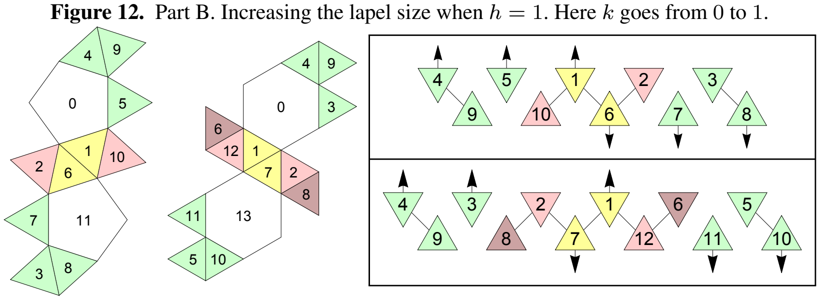


Figure 13. Part B. Increasing the lapel size when $h = 1$. Here k goes from 1 to 2.

Part C. Our table of order $n+1$ is now filled, except for the $(1,0)$ entry. For this we apply to each member of $\mathcal{S}_n$ a transformation that creates a corresponding, unique member of $\mathcal{S}_{n+1}(1,0)$. (Observe that in parts A and B of the proof, no members of $\mathcal{S}_{n+1}(1,0)$ were created.)

In this operation we take an arbitrary symmetric net and (1) cut it into halves by severing the neck at its middle edge, (2) to each half add an edge to increment the head size, (3) snap on an uncollared neck of size 1 to each half, and (4) reattach. See Figures 14 and 15 for examples. Note the arrow-gram, which gives a very clean and simple view, allowing a quick check with the arrow-gram criteria.

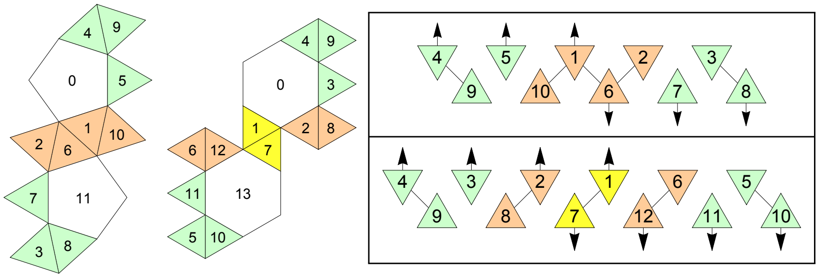


Figure 14. Part C. Splitting the neck. Here $(h, k) = (1, 1)$ becomes $(1, 0)$.

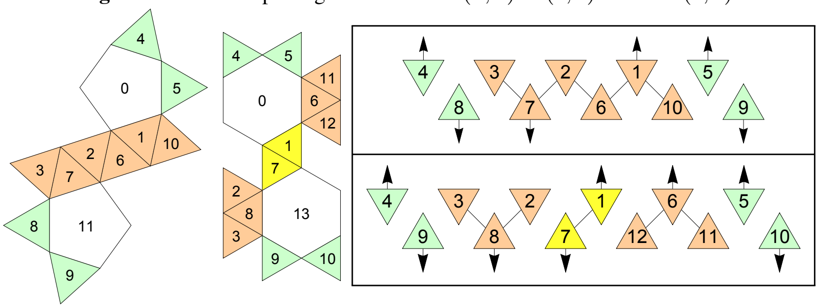


Figure 15. Part C. Splitting the neck. Here $(h, k) = (2, 1)$ becomes $(1, 0)$.

This procedure defines a bijection $f_C : \mathcal{S}_n \leftrightarrow \mathcal{S}_{n+1}(1, 0)$ (i.e., $f_C : \mathcal{S}_n \leftrightarrow \mathcal{S}(C_{n+1})$) and shows that $s_{n+1}(1, 0) = s_n$.

What we have so far demonstrated is that the table of order $n + 1$ is given by Figure 16.

$s_{n+1}(h,k)$	$h = 1$	2	3	4	$\cdots$	$n+1$
$k = 0$	s_n	$s_n(1,0)$	$s_n(2,0)$	$s_n(3,0)$	$\cdots$	$s_n(n,0)$
1	$s_n(1,0)$	$s_n(1,1)$	$s_n(2,1)$	$\cdots$	$s_n(n-1,1)$	
2	$s_n(1,1)$	$s_n(1,2)$	$\cdots$	$s_n(n-2,2)$		
$\vdots$	$\vdots$	$\vdots$	$\iddots$			
$n-1$	$s_n(1,n-2)$	$s_n(1,n-1)$				
n	$s_n(1,n-1)$					

Figure 16. The table of values of $s_{n+1}(h, k)$ established by parts A, B, C.

Finally, to employ the inductive hypothesis, we consult Figure 6 and directly substitute all the values for the $s_n(h, k)$ shown in Figure 16, except for $s_{n+1}(1, 0) = s_n$. But the inductive hypothesis also implies (as we have already shown in the proof of Theorem M1) that $s_n = F_{2n}$. The result is a table exactly like that of Figure 6, but of order $n + 1$, shown in Figure 7. The proof of the lemma is thereby finished.

Notice that our constructions recursively produce all the members of $\mathcal{S}_{n+1}$ from those of $\mathcal{S}_n$. Each symmetric net's evolution can be traced all the way back to the primordial case of $n = 1$, along with a corresponding sequence of mutations and metamorphoses of types A, B, and C.

Notice also that a bonus-identity is obtained by summing the diagonals in Figure 6 and applying (1), namely that $F_{2n} = n + \sum_{k=1}^{n} kF_{2(n-k)}$.

3. PATH NETS. Figure 17 shows the 27 path nets for $n = 5$, the first 8 of which are symmetric, each group sorted by neck size. There are $2(n-1)$ symmetric path nets for $n \geq 2$, which is easy to see by inspecting the symmetric cases—for each neck size $h = 1, 2, \ldots, n-2$ there are two possibilities, with one possibility each for $h = n-1$ and $h = n$.

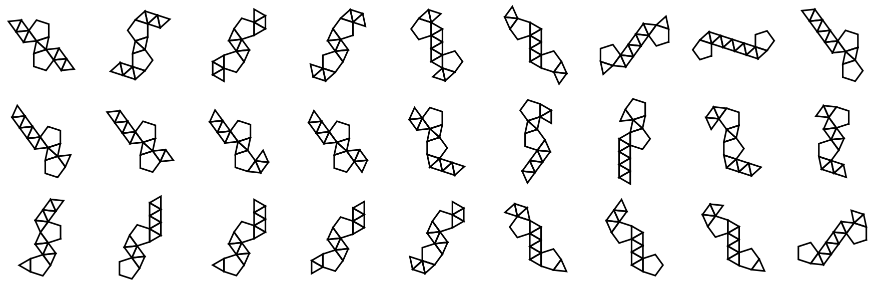


Figure 17. From Figure M24.

Looking at their arrow-grams makes this especially simple to see. In Figure 18 the neck is shown fully constructed with two other groups yet to be determined. In order to be the arrow-gram of a path there can only be at most two groups on either side of the neck, with arrows toward heads only allowed at the ends. To represent a symmetric path the two groups must be the same size (with no wrap-around) and the arrows toward heads must be at matching ends of the groups. In the figure this corresponds to either L_1 and R_1 or L_2 and R_2, giving precisely two choices because the group shown has size greater than 1. That occurs when $h \leq n-2$. When $h = n-1$ there is just one member of the group, so only one choice; when $h = n$ there are *zero* members available to form such groups, so again just one choice. Thus we get $2(n-2)+1+1 = 2(n-1)$ possibilities.

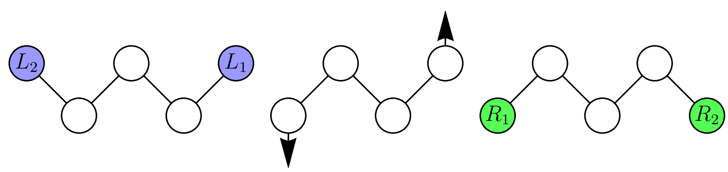


Figure 18. For symmetry, add arrows to heads at either L_1 and R_1 or L_2 and R_2.

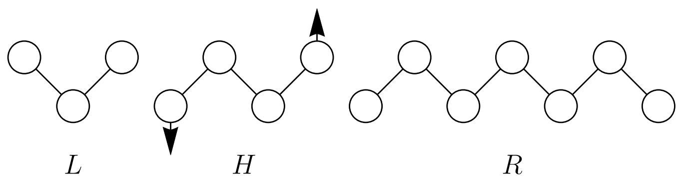
Claim. The number of path nets P_n of the n-antiprism is given by

$$P_n = \frac{3n^2 - 5n + 4}{2}.$$

To prove the claim, we first show that the number of such nets with neck size h is given by

$$P_n(h) = \begin{cases} 1 & \text{if} \quad h = n \\ 2 & \text{if} \quad h = n-1 \\ 3j-1 & \text{if} \quad h = n-j \quad (1 < j < n) \end{cases} \tag{1}$$

Proof. Let's build an antiprism path net like so: Start with a path in $\mathcal{G}_n$ consisting of all the $2n$ band members. Select any subpath H of length $2h$; this will be the neck. This leaves two other subpaths L and R on either side of H (where L or R or both may be empty), the whole band-path being the concatenation L-H-R. Remove the edges joining L and R to H and add the two heads to the thusly freed ends of H. Note that no lapels are possible in a path net — $k_1 = k_2 = 0$ — since a lapel would cause a head to have valence 3, so we'd no longer have a path. We'll view the result as a partial noncentered arrow-gram, for instance like so:

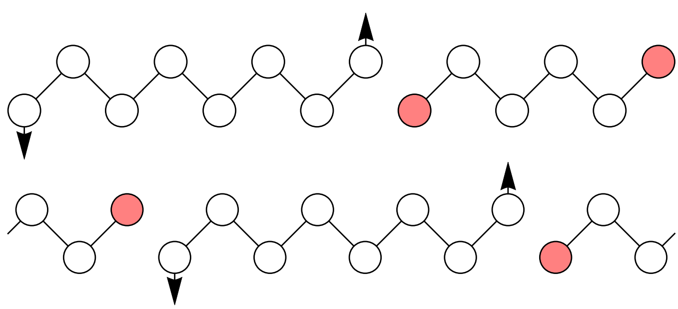


We let the lengths of L and R, respectively, be ℓ and $r = 2n - 2h - \ell$, taking $\ell \leq r$. Various types of configurations arise for varous choices of (ℓ, r), gathered as follows.

(00) 1 way: When $\ell = r = 0$, so $h = n$, then we have only the neck and its two heads, and there is only one such net for each n.

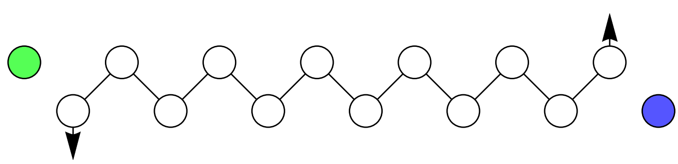
That's the first (obvious) case in equation (1).

(0+) 1 way: If $\ell = 0$ and $r > 0$, then L is empty and only R gets attached to a head. In this case, r must be even and the partial diagram shows that there is just one possibility, as can be seen when the diagram is neck-centered—the two possibilities for attaching a head (pink) give congruent diagrams:



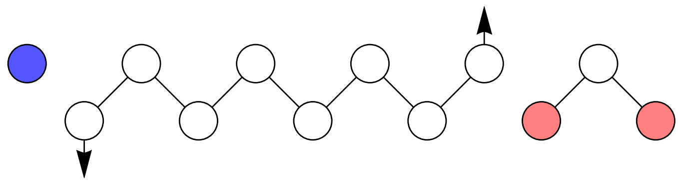
In all other cases we need to attach both the L and R groups to heads.

(11) 1 way: When $\ell = r = 1$, there is only one net for each n. The blue and green each get their own head:

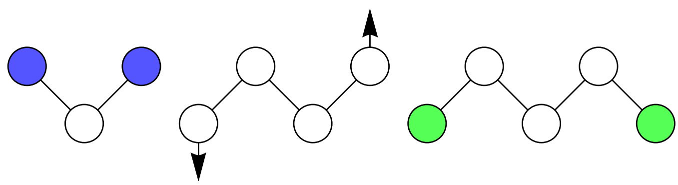


The result for $h = n - 1$ in (1) is now settled, since then $l + r = 2$, so the possibilities are $(l, r) = (0, 2)$ and $(l, r) = (1, 1)$. The cases **(0+)** and **(11)** show that there is just one net for each of those cases.

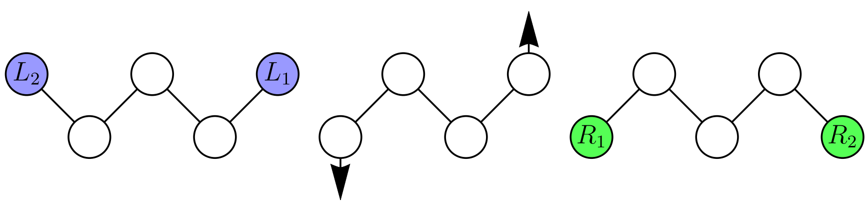
(1,1+) 2 ways: When $\ell = 1$ and $r > 1$, there are two nets for each n. Notice that r is odd; there is only one place (blue) for the L-head to go, but there are two (pink) for the R-head.



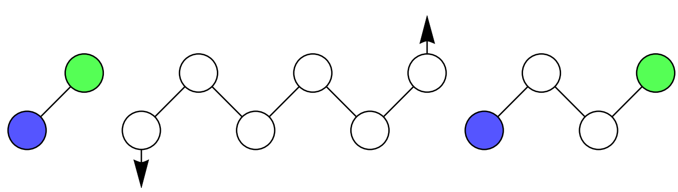
(odd≠odd) 4 ways: Assume $\ell > 1$ is odd and $\ell < r$. Because ℓ is odd, r must also be odd. Since both ℓ and r are odd, each of the ends of L are exposed to the same head, and both for R are exposed to the other head. Since $\ell \neq r$ there is no possibility of duplication (congruence). Thus there are two possibilities for each group, so four altogether.



(odd=odd) 3 ways: When $\ell > 1$ is odd and $\ell = r$, we have a symmetric situation. Although there are two positions for each head, as in the previous case, two of the four possibilities are congruent (namely, L_1-R_2 and L_2-R_1).



(even, even) 2 ways: Finally, when $l > 0$ is even, there are two possibilities. Since r will then be even (and positive), the two ends of each L and R are exposed to different heads, so we must choose one head for each (one green, one blue):



So now we can collect the results for values of $n - h$. Let $j = n - h$ and note that $\ell + r = 2j$, so ℓ takes on all the values in $\{0, 1, \ldots, j\}$. From what we gathered just previously, the number $P_n(l,r)$ of path nets with given (l,r) is

$$P_n(\ell,r)=\begin{cases}1 & \text{if} \quad \ell=0\\ 1 & \text{if} \quad \ell=r=1\\ 2 & \text{if} \quad \ell=1, r>1\\ 4 & \text{if} \quad \ell \text{ odd}, \ell<r\\ 3 & \text{if} \quad \ell \text{ odd}, \ell=r\\ 2 & \text{if} \quad \ell \text{ even.}\end{cases}$$

Thus we have (repeating ourselves a bit to gain momentum),

$$P_n(n)=P_n(0,0)=1,$$

$$P_n(n-1)=P_n(0,2)+P_n(1,1)=1+1=2,$$

$$P_n(n-2)=P_n(0,4)+P_n(1,3)+P_n(2,2)=1+2+2=5,$$

$$P_n(n-3)=P_n(0,6)+P_n(1,5)+P_n(2,4)+P_n(3,3)=1+2+2+3=8.$$

For j even and ≥ 4 we have the following (note the grouping in pairs),

$$\begin{aligned}P_n(n-j) &= P_n(0,2j)+P_n(1,2j-1)\\ &\quad +(P_n(2,2j-2)+P_n(3,2j-3))+\cdots\\ &\qquad +(P_n(j-2,j+2)+P_n(j-1,j+1))\\ &\quad +P_n(j,j)\\ &=1+2+(2+4)+\cdots+(2+4)+2\\ &=1+2+2+6\left(\frac{j}{2}-1\right)+2\\ &=3j-1,\end{aligned}$$

while for j odd and at least 5 we have

$$\begin{aligned}P_n(n-j) &= P_n(0,2j)+P_n(1,2j-1)+P_n(2,2j-2)\\ &\quad +(P_n(3,2j-3)+P_n(4,2j-4))+\cdots\\ &\qquad +(P_n(j-2,j+2)+P_n(j-1,j+1))\\ &\quad +P_n(j,j)\\ &=1+2+2+(4+2)+\cdots+(4+2)+3\\ &=1+2+2+6\left(\frac{j}{2}-\frac{3}{2}\right)+3\\ &=3j-1,\end{aligned}$$

and that verifies the claim in (1). Finally, summing over j gives the result:

$$P_n=\sum_{h=1}^{n}P_n(h)=\sum_{j=0}^{n-1}P_n(n-j)$$

$$= 1 + 2 + \sum_{j=2}^{n-1}(3j-1) = \frac{3n^2 - 5n + 4}{2}. \qquad \checkmark$$

4. HEAD COUNTS. In such a combinatorial undertaking as this, where are the binomial coefficients? In fact, they do make an appearance as the numbers $s_{n,j}$ of nets $N \in \mathcal{S}_n$ having j connected "head decorations" emanating from each head of N, i.e., paths of triangles starting at the head. (For example, in Figure 19, N_1 and N_2 have $j = 4$ and $j = 2$, respectively; in the latter those 2 decos on the left side are $\{3, 8\}$ and $\{7\}$.) It turns out that $s_{n,j} = \binom{n+j}{n-1-j}$, where $0 \leq j < n$. It must then be true (and it is well enough known; see [**2**]) that

$$\sum_{j=0}^{n-1} \binom{n+j}{n-1-j} = F_{2n}. \tag{2}$$

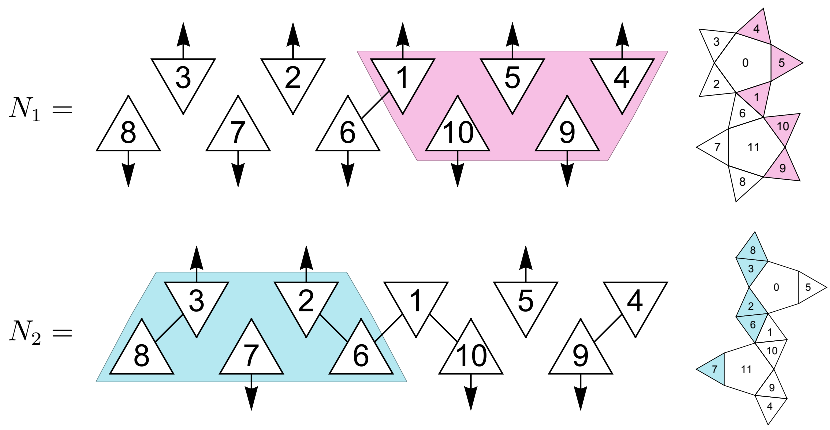


Figure 19. These are from Figure M20.

The "fine structure" is that for $r = h + k$, the number $s_{n,j}(r)$ of nets in $\mathcal{S}_n(h, k)$ having j decorations is $\binom{n-r-1+j}{n-r-j}$ for $1 \leq j \leq n - r$, while $s_{n,0}(n) = 1$. This can be proven by first noting that j can be associated with the length of an integer composition and using the fact that the number of compositions of m having length ℓ is given by

$$\sum_{(x_1, x_2, \ldots, x_\ell) \in \mathcal{C}_m} \prod_{i=1}^{\ell} x_i = \binom{m + \ell - 1}{m - \ell}.$$

See item (v) in [**2**], where this formula is proven using generating functions.

5. LABELED NETS. It was not one of our stated goals, but now that we have our counts, we can also calculate the number of labeled nets of a (labeled) antiprism. By labeling our antiprisms we simply make all the polygons *distinguishable*. Actually, since the polygons would be distinguishable if they were pairwise noncongruent, we could simply begin with such an antiprism and count the resulting set of noncongruent nets, rather than labeling. (We prefer to think of labeled versions, however, because we are also interested in the symmetric nets, of which there would be none in such a maximally lopsided scenario. Also, it turns out that we could have done everything, including the symmetric counts, if we used antiprisms whose bands have congruent nonequilateral isosceles triangles instead of equilateral ones. That would even make the situation of $n = 3$ more natural. There is no combinatorial difference, just an aesthetic one, at most.)

For each asymmetric net N (in $\mathcal{T}_n \setminus \mathcal{S}_n$) there are $4n$ distinct labelings (starting with a fixed labeling of the antiprism itself, such as ours). This follows from the well-known fact that the (full) *symmetry group* of the antiprism has order $4n$, but to see directly why this is so, one can count distinct symmetric motions on the antiprism. (The following descriptions of the symmetry group might be unnecessary, but are included for lack of a single reference.)

First, we observe that the antiprism has n distinct symmetries by rotation—call them R_k—through an angle $2k\pi/n$ ($0 \le k < n$) about an axis v through the centers of its two n-gons. (The *identity* element is R_0.) If the antiprism is labeled, then each R_k permutes the labels in a unique way. As an example, the two congruent nets in Figure 12 are obtained from one another by rotations of an angle $\pi/2$ about v when applied to the 4-antiprism; refer to Figure 9.

Another rotational symmetry F_1 is the result of first rotating the antiprism through a half-revolution about a line parallel to the n-gons and through the center of the antiprism, and second, if (and only if) n is even, rotating the result by π/n about the aforementioned axis v. It is clear that F_1 flips the top and bottom n-gons, and the n resulting compositions $F_1 \circ R_k$ give n symmetries, all distinct from the R_k. If F_0 represents the identity, then the set $\{F_j \circ R_k : 0 \le j \le 1 \text{ and } 0 \le k < n\}$ represents the $2n$ members of the *rotation group* (often denoted D_n) on the antiprism.

Finally, for each of these $2n$ *proper* symmetries there is an *improper* symmetry—one involving an *inversion*, which is a point-reflection through its center. Performing an inversion followed again by a rotation of π/n about v when n is even, results in a new symmetry I_1 that results in a reversal of the orientation of the labels about the band. If I_1 represents this symmetry and I_0 is the identity, this gives the full symmetry group $D_{nd} = \{I_i \circ F_j \circ R_k : 0 \le i \le 1,\ 0 \le j \le 1, \text{ and } 0 \le k < n\}$, which has $4n$ distinct members, each giving rise to a distinct labeling on an (asymmetric) net $N \in \mathcal{T}_n \setminus \mathcal{S}_n$.

A (symmetric) net $N \in \mathcal{S}_n$ has $2n$ distinct labelings, since any labeling of such a net is unaffected by I_1.

So if ℓ_n is the total number of labeled nets, then we have

$$\begin{aligned}
\ell_n &= 4n(t_n - s_n) + 2ns_n \\
&= 4nt_n - 2ns_n \\
&= 4n(s_n(s_n + 1)/2 - 2ns_n \\
&= 2n(s_n)^2 \\
&= 2n(F_{2n})^2.
\end{aligned}$$

The number ℓ_n can be obtained, at least for any given fixed n, using Kirchhoff's matrix-tree theorem [**1**, p. 203], which says that the number of labeled trees of a graph is any minor of $D - A$, where A is the adjacency matrix of the graph and D is the diagonal matrix of its valences. As an example, for the graph $\mathcal{G}_5$ we have the following, where the first row/column corresponds to the face labeled 0 in our scheme. In general, we will have a similar $(2n+2)\times(2n+2)$ matrix for each n—resize appropriately and replace each 5 in the following matrix with n.

$$D-A=\begin{pmatrix}
5 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 3 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0\\
-1 & 0 & 3 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0\\
-1 & 0 & 0 & 3 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & -1 & -1 & 0 & 0\\
-1 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & -1 & -1 & 0\\
0 & -1 & -1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & -1\\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & -1\\
0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 3 & 0 & 0 & -1\\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 3 & 0 & -1\\
0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 3 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 5
\end{pmatrix}$$

Presumably one can somehow directly compute the minors of such matrices for variable n, but the author has not succeeded at that. We do know indirectly, though, by virtue of our results (and Kirchoff's theorem), that they are equal to $\ell_n = 2n(F_{2n})^2$.

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