A Five-fold Flimflam

Rick Mabry (rmabry [AT] lsus [DOT] edu), LSU in Shreveport, Shreveport, LA 71115-2399

A teacher I know recently mentioned to me something that aroused first my interest, then my suspicions, then my surprise. The teacher had found me in my office folding paper and cutting out snowflakes, thereby fulfilling her suspicions of what my job entails. (Hey, I don't often sit around making paper cutouts, but it was near the holiday season and as a math club advisor I wanted something mathematically festive for the ..., oh, never mind.) I proudly showed off my artwork, "See the nice hexagonal symmetry? Crystallographers call it $p6m$." She examined me for a moment, then looked at the few unfolded flakes and at the floor covered with white paper trimmings. She admitted they were rather nice flakes, and said I could have a gold star. I beamed. Then she told me about a talk she had attended at a recent NCTM conference (she was a middle school math teacher), in which it was demonstrated how to make snowflakes having *five-fold symmetry*. "But ... snowflakes don't have five-fold symmetry," I explained. She retracted her offer of a gold star.

Now, perhaps like most of you, I was accustomed to making snowflakes with triangular or hexagonal radial symmetries (never the insipid octagonal symmetries seen in department stores during the holidays!), and I had never even suspected it might be possible to make five-fold symmetric figures from paper cutting. In fact, as I thought about it, it seemed impossible, or at least difficult. I asked if there was some strange trick about it. Did you have to measure anything? Was it hard to do? No, was the answer, it was all done using midpoints and folds on a piece of notebook paper, in a similar manner to the usual hexagonal versions. Well, I thought, perhaps it was indeed possible, since pentagons can be constructed using a compass and straightedge, but I couldn't imagine how that would extend to paper folding. I was promised a copy of the hand-outs given during the talk, so I could see it for myself.

And so, I learned how to create the snowflake, *using an ordinary* $8\frac{1}{2}'' \times 11''$ *sheet of paper*. (Fie!) First, with the the sheet oriented with its long side horizontal, fold the left side to the right and crease. The crease is now the left side. Mark the middle (M in fig. 1) of the top of the folded sheet (note: a "legal" operation — this can be done by folding the top in half and making a very short crease). Fold the bottom-left corner C up to M and crease (the crease is QR in fig. 2 and fig. 3). Fold the top flap ($MQRD'$ in fig. 3) by bisecting angle MQR, and then fold the triangular flap TQM to the back about QM. Finally, cut off the "excess" to suit your fancy, so that the inside folds of the folded flake are all the same, and start cutting your designs. You will unfold it to find a fine five-fold flimflake.

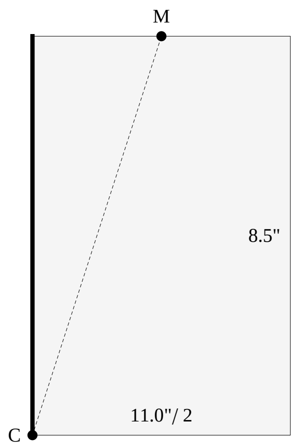

Fig. 1

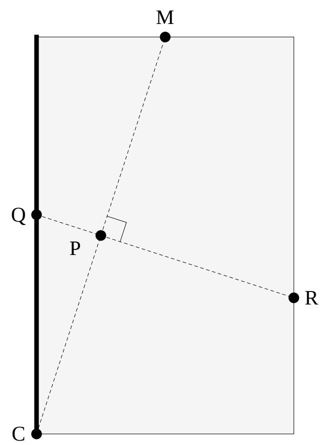

Fig. 2

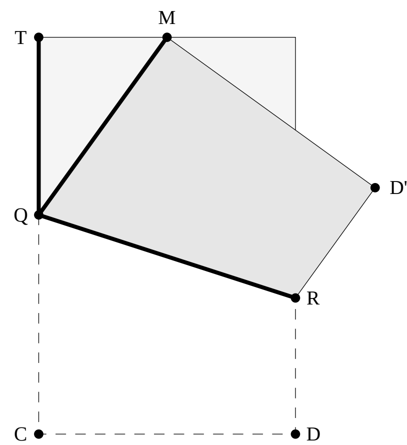

Fig. 3

Something very amusing about this is the closeness to which the symmetry is approximated. In fact, the measure of the angle $\theta = \angle TQM$ is within about one-seventh of a degree of the 36° required for perfect symmetry. (Heck, given that I myself cannot fold to within that tolerance, I may as well consider the symmetry perfect.) To see this, simply let $a = 8.5$, $b = 11.0$, and it is easy to show (assign it to your trig class) that

$$\theta = \tan^{-1} \frac{8ab}{16a^2 - b^2} \approx 35.856°,$$

an accuracy of about 0.4%.

For homework: Find how few of your fellow faculty fall for the folderol.

Added June 15, 2008. The pages above have been very slightly edited from the original (posted Sep. 2, 1998) — mainly to remove reference to a nonexistant Figure 4, to marginally improve the existing Figures 1 through 3, and to remove the names (of human figures) to protect the innocent. It could be made clearer still but I do not wish to alter it further, as there is a link to it in the article "Going for the stars" (FFF #151), *The College Mathematics Journal*, **30**, no. 5 (1999), p. 383. Any further changes will be made below this space.

I do not know the origin of the flim-fold trick, but located a copy of the instructions I saw; a photocopy is at the following link.

http://www.lsus.edu/sc/math/rmabry/folding/fivefold-LATM1996-p6.jpg

The following page (as of today) achieves the same thing:

`http://www.origami-resource-center.com/five-pointed-star.html`

More interesting is the "Betsy Ross" model, which uses paper of size $8\frac{1}{2}'' \times 10''$.

`http://www.ushistory.org/betsy/flagstar.html`

Hmm, something about the history presented there smells fishy, similar to George Washington and the cherry tree. (Washington may not have told a lie but others probably told lies about him.)

http://xroads.virginia.edu/%7Ecap/gw/gwmoral.html

Aside from history, an interesting thing about the Betsy Ross construction is its accuracy. As mentioned earlier, the pertinent angle produced in the "Five-fold Flimflam" is $\angle TQM$, which is about $\approx 35.856°$. But what should have been mentioned is that this angle, part of the final "back flap" in the construction, is not the only angle under consideration — the other four reprentatives of the central angles in the five-fold flake arise from $\frac{1}{2}\angle MQR$, which turns out to be about $36.036°$. The maximum error comes from $\angle TQM$, about $0.144°$. On the other hand, the central angles in Betsy's construction are one of $\beta = \sin^{-1}(b/2a)$, with $a = 8.5''$ and $b = 10''$, or $\beta \approx 36.032°$, and four pf $(180° - \beta)/4 \approx 35.992°$. Thus (if her folding is so good), Betsy's error is only $.032°$, which is almost a five-fold improvement over the flimflammers' fudging.

In any case, the thing I find strangest about both of the constructions is the final cut — it is arbitrary. That's fine for five-fold snow(job)flakes, but does not seem sufficient for the stars on the Stars and Stripes. As shown on the page, "5-Pointed Star in One Snip", linked above, one obtains varying "stars" depending on the "angle" of the "one snip". Where should one cut to obtain a perfect pentagram? Shouldn't Betsy have been a bit more specific if she's suggesting mass production of the things??
In digging further, I see there is some reason to be skeptical about the Betsy Ross tale, but maybe it is true.

http://en.wikipedia.org/wiki/Betsy_Ross_flag

I'll look into it further...