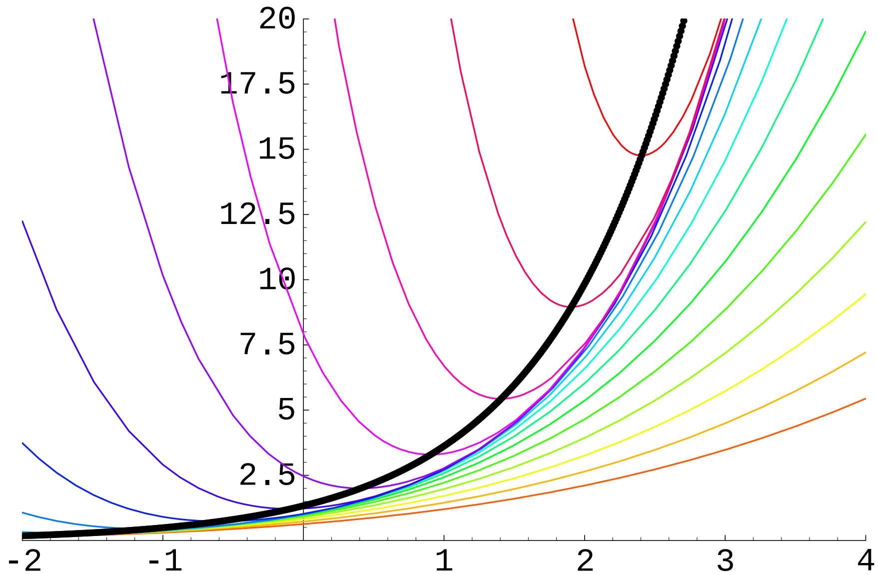
Minima of even Taylor polynomials of the exponential

Proposed by Rick Mabry, LSUS, Shreveport, LA, rmabry@lsus.edu

Let n be a fixed positive even integer and let $P_a(x)$ denote the Taylor polynomial of degree n of the function e^x, centered at $x = a$. After proving that $P_a(x)$ has a single minimum and letting $x = t_a$ denote where this minimum occurs, prove that the parameterized curve $\{(t_a, P_a(t_a)) : a \in \mathbb{R}\}$ is itself the graph of an exponential function.

[Dear Editor: I would have thought that this is surely not new, but I cannot find it. Apologies if it's been done. Here's a picture (for $n = 4$), in case my description is less than enticing, Taylor polynomials colored, the paramaterized curve in black. — *RM*]



Solution. First we'll show that $P_a(x)$ is positive with a single critical point. [Surely this has been observed thousands of times.]

We can show that the (even) Taylor polynomials of e^x are strictly positive, by showing that $P_0(x)$ is positive, since $P_a(x) \equiv e^a P_0(x-a)$. All the terms of $P_0(x)$ are positive for $x > 0$, so we consider $x < 0$. But choosing the $\xi \in (x, 0)$ guaranteed by by Taylor's theorem, we have

$$e^x = P_0(x) + e^{\xi} \frac{x^{n+1}}{(n+1)!} < P_0(x),$$

since $x^{n+1} < 0$ (and since $n+1$ is odd). It is now clear that for $x \leq 0$, we have $P_0(x) \geq e^x > 0$.

Now it will be clear that each such even Taylor polynomial has a unique minimum, as it follows inductively from the fact that the second derivative of such a polynomial is another such polynomial of two degrees fewer. (It is easy to show it is true for $n = 2$ and we omit those details.) Therefore, our polynomials are all strictly concave up and positive and so must have unique minima.

Now for the main result. Any minimum of $P_a(x)$ must occur where $P_a'(x)$ is zero. But $P_a'(x)$ is such that

$$P_a(x) = \frac{e^a}{n!}(x-a)^n + P_a'(x).$$

Since $P_a'(t_a) = 0$, we have

$$P_a(t_a) = e^a \frac{(t_a - a)^n}{n!}.$$

But now letting r be the unique zero of P_0', we have

$$P_a(t_a) = \frac{r^n}{n!} e^a.$$

It is also evident that $r = t_a - a$, so we have

$$P_a(t_a) = \frac{r^n}{n!} e^{t_a - r} = \left(\frac{r^n}{e^r n!} \right) e^{t_a}.$$

Thus we have that the parametric plot of graph of $\{(t_a, P_a(t_a)) : a \in \mathbb{R}\}$ can be reparameterized as $\{(x, y) : y = Ce^x\}$, where $C = \dfrac{r^n}{e^r n!}$ is a constant depending only on n.

The only final quibble would be as to the domain of the parameterization. But since $t_a = r + a$, we see that t_a takes all real values.