Intersecting Curves: 10712
Author(s): Paul Deiermann and Rick Mabry
Source: *The American Mathematical Monthly*, Vol. 107, No. 10 (Dec., 2000), p. 958
Published by: Mathematical Association of America
Stable URL: http://www.jstor.org/stable/2695606
Accessed: 28/09/2011 22:39

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at
http://www.jstor.org/page/info/about/policies/terms.jsp

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Mathematical Association of America is collaborating with JSTOR to digitize, preserve and extend access to *The American Mathematical Monthly*.

http://www.jstor.org

Note that $a \sin B = b \sin A$ by the law of sines. Let $4L = a^2 \sin^2 B = b^2 \sin^2 A$ and consider the function $h(x) = x + L/x$. A computation shows that $h(x) = h(y)$ if and only if $(x-y)(L-xy) = 0$. Thus, for $x(u) = b\cos^2\alpha - u$ and $y(u) = a\cos^2\beta - f(u)$, equality of the two expressions (1) and (2) implies that either $x(u) = y(u)$ or $x(u)y(u) = L$. Note that $x(u) = y(u)$ means that $u - f(u) = b\cos^2\alpha - a\cos^2\beta$ and, since the function on the left is increasing, this equality holds for at most one value of u. For all other values of u, the equality $x(u)y(u) = L$ holds, so it holds for all u in I_A by continuity. Thus $(b\cos^2\alpha - u)(a\cos^2\beta - f(u)) = L$ for all u in I_A, which implies that f is a linear fractional transformation.

Write $f_{A,B}$ for f, and use the natural analogies with the definition of $f_{A,B}$ to define $f_{B,C} : I_B \to I_C$ and $f_{C,A} : I_C \to I_A$. Set $g = f_{C,A} \circ f_{B,C} \circ f_{A,B}$. Since $g \circ g$ is a composition of linear fractional transformations, it is itself a linear fractional transformation. On the other hand, g is a decreasing homeomorphism of I_A, and so it has a unique fixed point x in I_A. Also, $g(0) = u_A$ and $g(u_A) = 0$. Thus 0, x, and u_A are three distinct fixed points of $g \circ g$. A linear fractional transformation with three distinct fixed points is the identity, so $g \circ g$ is the identity, as desired.

Editorial comment. This problem appeared also as part (**a**) of problem E3236 in this MONTHLY [1987, 877; 1990, 529], where partner circles were called *companion incircles*.

Solved also by R. Choisuren (Mongolia), J. Duncan & S. Tabachnikov, D. Einfeld, S. B. Ekhad, J. Fukuta (Japan), S. Haas & A. Bliss, J. Lee, J. C. Linders (The Netherlands), J. H. Lindsey II, O. P. Lossers (The Netherlands), C. R. Pranesachar (India), V. Schindler (Germany), R. Stong, I. Talata, A. Tissier (France), and GCHQ Problems Group (U. K.).

REVIVALS

Intersecting Curves

10712 [1999, 166; 2000, 463]. *Proposed by Paul Deiermann, Lindenwood University, St. Charles, MO, and Rick Mabry, Louisiana State University, Shreveport, LA.* Let $f(x)$ and $g(y)$ be twice continuously differentiable functions defined in a neighborhood of 0, and assume that $f(0) = 1$, $g(0) = f'(0) = g'(0) = 0$, $f''(0) < 0$, and $g''(0) > 0$.
(**a**) For sufficiently small $r > 0$, show that the curves $x = g(y)$ and $y = rf(x/r)$ have a common point (x_r, y_r) in the first quadrant with the property that, if (x, y) is any other common point, then $x_r < x$.
(**b**) Let $(t_r, 0)$ denote the x-intercept of the line passing through $(0, r)$ and (x_r, y_r). Show that $\lim_{r\to 0+} t_r$ exists, and evaluate it.
(**c**) Is the continuity of f'' and g'' a necessary condition for $\lim_{r\to 0+} t_r$ to exist?

Editorial comment. Several solvers argued in part (**c**), even without assuming the continuity of g'', that the function g must be increasing on some interval $[0, b]$. The editorial comment published with the solution provided a putative counterexample to that assertion, and the solvers who argued in this way were deemed to have provided an incorrect solution. In fact, the solvers were correct and the editors erred. Kenneth Schilling provides the following simple argument defending the claim.

Proposition. If g is a twice differentiable function defined in a neighborhood of 0, and if $g(0) = g'(0) = 0$ and $g''(0) > 0$, then g is increasing on $[0, b]$ for some $b > 0$.

Proof. Since $g''(0) > 0$, we have $\left(g'(y) - g'(0)\right)/y = g'(y)/y > 0$ for all sufficiently small $y > 0$. Hence $g'(y) > 0$ on $(0, b]$ for some b. It follows that g is increasing in this interval, and since g is continuous at 0, g is increasing on $[0, b]$. □

Here is the corrected list of solvers of the original problem.

Solved by R. J. Chapman (U. K.), J. H. Lindsey II, O. P. Lossers (The Netherlands), A. Nijenhuis, K. Schilling, A. Tissier (France), and the proposer.