10807. *Proposed by Marc Deléglise, Université Lyon, Lyon, France.* For positive parameters u and v, evaluate

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \sqrt{1 + 4^n u^{2k} v^{2n-2k}}.$$

10808. *Proposed by Enrico Valdinoci, University of Texas, Austin, TX.* Prove that the series $\sum_{n=0}^{\infty} (\cos(nx))^{n^r}$ diverges for all $x \in \mathbb{R}$ if $r \le 2$ but converges for almost every $x \in \mathbb{R}$ with respect to Lebesgue measure if $r > 2$.

SOLUTIONS

Intersecting Curves

10712 [1999, 166]. *Proposed by Paul Deiermann, Lindenwood University, St. Charles, MO, and Rick Mabry, Louisiana State University, Shreveport, LA.* Let $f(x)$ and $g(y)$ be twice continuously differentiable functions defined in a neighborhood of 0, and assume that $f(0) = 1$, $g(0) = f'(0) = g'(0) = 0$, $f''(0) < 0$, and $g''(0) > 0$.

(a) For sufficiently small $r > 0$, show that the curves $x = g(y)$ and $y = rf(x/r)$ have a common point (x_r, y_r) in the first quadrant with the property that, if (x, y) is any other common point, then $x_r < x$.

(b) Let $(t_r, 0)$ denote the x-intercept of the line passing through $(0, r)$ and (x_r, y_r). Show that $\lim_{r\to 0+} t_r$ exists, and evaluate it.

(c) Is the continuity of f'' and g'' a necessary condition for $\lim_{r\to 0+} t_r$ to exist?

Solution by Alain Tissier, Montfermeil, France. The conclusions in **(a)** and **(b)** remain correct even if we do not assume continuity of f'' and g''. We retain only the continuity of the first derivative and the existence and sign of f'' and g'' at zero. We prove a generalization, weakening the hypotheses as follows: Assume that f is a continuous mapping on $[0, a]$ with $a > 0$ and that $f(x) = 1 - \lambda x^p + o(x^p)$ as $x \to 0$ for some $p > 0$ and $\lambda > 0$. Assume also that g is a continuous mapping on $[0, b]$ with $b > 0$ and that $g(y) = \mu y^q + o(y^q)$ as $y \to 0$ for some $q > 1$ and $\mu > 0$. The conditions on f and g in the problem statement imply these hypotheses with $p = q = 2$, $\lambda = -f''(0)/2$, and $\mu = g''(0)/2$.

(a) With a and b sufficiently small, we may suppose $f(x) > 0$ on $[0, a]$ and $g(y) > 0$ on $(0, b]$. Let $m > 0$ be the maximum of $f(x)$ on $[0, a]$. For each $r > 0$, let $f_r(x) = rf(x/r)$. Then f_r is a continuous mapping on $[0, ra]$, $f_r(x) = r - \lambda r^{1-p} x^p + o(x^p)$, and the maximum of f_r on $[0, ra]$ is mr. Assume that $r \le b/m$. Then $f_r(x) \le b$ on $[0, ra]$.

The function h_r defined by $h_r(x) = g(f_r(x)) - x$ is defined and continuous on $[0, ra]$, and it satisfies $h_r(0) = g(r) > 0$ and $h_r(ra) = g(rf(a)) - ra$. Since $g(rf(a)) = O(r^q)$ as $r \to 0$ and since $q > 1$, we have $g(rf(a)) = o(r)$. Hence there exists $\delta > 0$ so that $h_r(ra) \le 0$ if $r < \delta$. Assume that $r \le \delta$. The function h_r is continuous on $[0, ra]$, $h_r(0) > 0$, and $h_r(ra) \le 0$, so by the intermediate value theorem there exists $x_r > 0$ such that $h_r(x_r) = 0$ and $h_r(x) > 0$ on $[0, x_r)$. The curves $y = f_r(x)$ and $x = g(y)$ have a common point (x_r, y_r) with $y_r = f_r(x_r)$ and $x_r = g(y_r)$, and every other common point has a larger x-coordinate.

(b) We show that, in our more general setting, a finite nonzero limit exists if and only if $1/p + 1/q = 1$, and then the limit is $1/(\lambda\mu^{p-1})$. Since $0 < x_r \le ra$, we have $x_r = O(r)$ as $r \to 0$. Hence $y_r = r - \lambda r^{1-p} x_r^p + o(x_r^p) = O(r)$ as $r \to 0$. We may use this to obtain $x_r = g(y_r) = \mu y_r^q + o(y_r^q) = O(r^q)$ and $y_r = r - \lambda r^{1-p} x_r^p + o(x_r^p) = r + O(r^{1-p+pq})$ as $r \to 0$. This in turn leads to the further refinement $x_r = \mu r^q + o(r^q)$ and $y_r =$

$r - \lambda\mu^p r^{1+p(q-1)} + o(r^{1+p(q-1)})$ as $r \to 0$. An easy computation gives $t_r = rx_r/(r - y_r)$, so

$$t_r = \frac{\mu r^{q+1}}{\lambda\mu^p r^{1+p(q-1)}} + o(r^{p+q-pq}) = \frac{r^{p+q-pq}}{\lambda\mu^{p-1}} + o(r^{p+q-pq}).$$

Thus $\lim_{r\to 0+} t_r$ is finite and nonzero if and only if $p + q - pq = 0$, which may be written $1/p + 1/q = 1$. In this case, we have $\lim_{r\to 0+} t_r = 1/(\lambda\mu^{p-1})$. With the hypotheses of the original problem, the limit is $-4/(f''(0)g''(0))$.

Editorial comment. Several contributors asserted incorrectly in part **(c)** that g must be increasing in some interval $[0, b]$. The example $g(y) = y^2 + 2y^4\sin(y^{-2})$ with $g(0) = 0$ shows that this is false. The special case $f(x) = \sqrt{1-x^2}$ and $g(y) = 1 - \sqrt{1-y^2}$ is Problem 5 in J. D. E. Konhauser, D. Velleman, and S. Wagon, *Which Way Did the Bicycle Go?*, MAA, Washington, DC, 1996.

Solved also by A. Nijenhuis and the proposer.

A Cute Characterization of Acute Triangles

10713 [1999, 166]. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.* Given a triangle with angles $A \ge B \ge C$, let a, b, and c be the lengths of the corresponding opposite sides, let r be the radius of the inscribed circle, and let R be the radius of the circumscribed circle. Show that A is acute if and only if $R + r < (b + c)/2$.

Solution by Heinz-Jürgen Seiffert, Berlin, Germany. The condition $A \ge B \ge C$ may be weakened to $A \ge |B - C|$. The circumradius R satisfies $2R = a/\sin A = b/\sin B = c/\sin C$, so $A + B + C = \pi$ implies

$$\frac{b+c}{2} = R(\sin B + \sin C) = 2R\sin\frac{B+C}{2}\cos\frac{B-C}{2} = 2R\cos\frac{A}{2}\cos\frac{B-C}{2}.$$

The inradius is $r = 4R\sin(A/2)\sin(B/2)\sin(C/2)$, so

$$\begin{aligned} r &= 2R\sin\frac{A}{2}\left(\cos\frac{B-C}{2} - \cos\frac{B+C}{2}\right) = 2R\sin\frac{A}{2}\left(\cos\frac{B-C}{2} - \sin\frac{A}{2}\right) \\ &= R\left(2\sin\frac{A}{2}\cos\frac{B-C}{2} - \sin^2\frac{A}{2} + \cos^2\frac{A}{2} - 1\right). \end{aligned}$$

Thus,

$$R + r - \frac{b+c}{2} = R\left(\sin\frac{A}{2} - \cos\frac{A}{2}\right)\left(2\cos\frac{B-C}{2} - \sin\frac{A}{2} - \cos\frac{A}{2}\right).$$

But

$$\cos\frac{B-C}{2} - \sin\frac{A}{2} = \cos\frac{B-C}{2} - \cos\frac{\pi - A}{2} = 2\sin\frac{B}{2}\sin\frac{C}{2},$$

so we have

$$R + r - \frac{b+c}{2} = R\left(\sin\frac{A}{2} - \cos\frac{A}{2}\right)\left(2\sin\frac{B}{2}\sin\frac{C}{2} + \cos\frac{B-C}{2} - \cos\frac{A}{2}\right).$$

The condition $A \ge |B - C|$ implies that $\cos\big((B - C)/2\big) \ge \cos(A/2)$, so the last factor on the right-hand side is positive. It follows that $R + r - (b + c)/2 < 0$ if and only if $\sin(A/2) < \cos(A/2)$, which occurs if and only if A is acute.

Solved also by Z. Ahmed & M. A. Prasad (India), S. Amighbech (France), S. Andras (Romania), J. Anglesio (France), R. J. Chapman (U. K.), D. Donini (Italy), J. Fukuta, M Hajja (U. A. E.), N. Heideman (South Africa), A. Kalakos (Greece), M. S. Klamkin (Canada), J. H. Lindsey II, O. P. Lossers (The Netherlands), G. Peng, J. S. Robertson & J. Rob, V. Schindler (Germany), I. Sofair, T. V. Trif (Romania), GCHQ Problems Group (U. K.), and the proposer.