Proof of Fact 5. Holding θ and r constant, let $f(y) = x(y, \theta, r)$ and differentiate (A) and (B) with respect to y to get

$$f'(y) = -\tan\alpha + (1-y)\sec^2\alpha \frac{d\alpha}{dy}$$

and

$$\frac{d\alpha}{dy} = \frac{\sin\theta}{2r}\sec(\theta+\alpha).$$

Differentiating each once more gives

$$f''(y) = -2\sec^2\alpha \frac{d\alpha}{dy} + (1-y)\sec^2\alpha\left(2\tan\alpha\left(\frac{d\alpha}{dy}\right)^2 + \frac{d^2\alpha}{dy^2}\right)$$

and

$$\frac{d^2\alpha}{dy^2} = \frac{\sin\theta}{2r}\sec(\alpha+\theta)\tan(\alpha+\theta)\frac{d\alpha}{dy}.$$

Combining these gives

$$\frac{d^2x}{dy^2} = \frac{d\alpha}{dy}(\sec^2\alpha)(-2+\mathcal{J}),$$

where

$$\mathcal{J} = (1-y)\frac{\sin\theta}{2r}\sec(\alpha+\theta)(2\tan\alpha + \tan(\alpha+\theta)).$$

We'll show that $\mathcal{J} < 3/2$, which will prove that $f''(y) < 0$.
Since $\tan\alpha < \tan(\alpha+\theta)$, we have

$$\begin{aligned}
\sec(\alpha+\theta)(2\tan\alpha + \tan(\alpha+\theta)) &< 3\sec(\alpha+\theta)\tan(\alpha+\theta) \\
&= \frac{3\sin(\alpha+\theta)}{1-\sin^2(\alpha+\theta)} \\
&= \frac{3(\sin\theta)(1+\frac{y}{2r})}{1-(\sin^2\theta)(1+\frac{y}{2r})^2}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\mathcal{J} &< \frac{(1-y)}{2r}\,\frac{3\left(\frac{2r}{1+2r}\right)^2\left(1+\frac{y}{2r}\right)}{1-\left(\frac{2r}{1+2r}\right)^2\left(1+\frac{y}{2r}\right)^2} \\
&= \frac{3(2r+y)}{1+4r+y} \\
&< \frac{3}{2}.
\end{aligned}$$