**Proof of Fact 6.** $f(y)$ has a unique maximum for $0 \le y \le 1$, since $f(0) = f(1) = 0$ and $f''(y) < 0$ on this interval. By showing $f'(1/2) > 0$, it will follow that this maximum occurs for $1/2 < y < 1$.

We've already noted (implictly) the dependence of $\alpha$ on $y$, but let's set $y = 1/2$ in (B) to get

$$\sin(\alpha+\theta) = (\sin\theta)\left(1 + \frac{1}{4r}\right),$$

and continue to write $\alpha$ for the specific value of $\alpha$ so obtained (which still depends on the fixed values of $\theta$ and $r$). Using the fact that $\sec^2\alpha > \sec\alpha$, our formula for $f'(y)$ from a previous napkin gives

$$\begin{aligned} f'(1/2) &> -\tan\alpha + \frac{1}{4r}\sec\alpha\sin\theta\sec(\alpha+\theta) \\ &= \sec\alpha(-\sin\alpha + (\sin(\alpha+\theta) - \sin\theta)\sec(\alpha+\theta)) \\ &= \sec\alpha\sec(\alpha+\theta)\cdot\mathcal{L}, \end{aligned}$$

where $\mathcal{L} = \sin(\alpha+\theta) - \sin\alpha\cos(\alpha+\theta) - \sin\theta$. We're done if $\mathcal{L} > 0$.

We note that $\mathcal{L} = 0$ for $\alpha = 0$, so we'll be done if $\dfrac{\partial\mathcal{L}}{\partial\alpha} > 0$. And indeed,

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\alpha} &= \cos(\alpha+\theta) - \cos\alpha\cos(\alpha+\theta) + \sin\alpha\sin(\alpha+\theta) \\ &> \cos(\alpha+\theta) - \cos(\alpha+\theta) + \sin\alpha\sin(\alpha+\theta) \\ &= \sin\alpha\sin(\alpha+\theta) \\ &> 0. \end{aligned}$$