

Proof of Fact 7. We want to show that $f'(0) < -f'(1)$. Letting $a = 1/(2r)$, this translates into proving that $a \tan \theta < \tan((\sin^{-1}((a+1)\sin \theta) - \theta))$, or

$$\tan^{-1}(a \tan \theta) + \theta < \sin^{-1}((a+1)\sin \theta). \quad (\text{C})$$

Note that $a > 1$ and that $(a+1)\sin \theta > 1$ (our condition of the maximum angle). Both sides of the inequality in (C) are zero for $\theta = 0$, so we are finished if the inequality holds when differentiated. That is, we are done if we can show that

$$1 + \frac{a \sec^2 \theta}{1 + a^2 \tan^2 \theta} < \frac{(a+1) \cos \theta}{\sqrt{1 - (a+1)^2 \sin^2 \theta}}. \quad (\text{D})$$

Squaring both sides, cross-multiplying, then gathering everything to the right (brute-force here; I won't say if I had any electronic assistance), our inequality in (D) is true if

$$a^2 \sin^2 \theta (3 - (a^2 + 2a + 3) \sin^2 \theta) > 0.$$

Again using the fact that $\sin \theta < \frac{1}{a+1}$, we have

$$\begin{aligned} 3 - (a^2 + 2a + 3) \sin^2 \theta &> 3 - \frac{a^2 + 2a + 3}{(a+1)^2} \\ &= \frac{2a(a+2)}{(a+1)^2}, \end{aligned}$$

which is positive, so we're done.

