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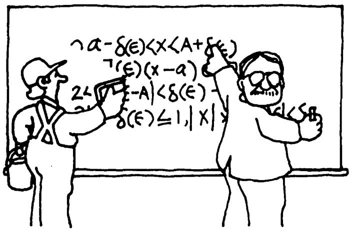
PROBLEMS AND SOLUTIONS



This section contains problems that challenge students and teachers of college mathematics. We urge you to participate actively in this section by submitting solutions to the published problems and by proposing problems that are new and interesting. In the interests of maintaining the variety of problems published, the editors are especially interested in receiving problem proposals that span the entire undergraduate curriculum.

Whenever possible, a proposed problem should be accompanied by a solution, appropriate references, and any other material that would be helpful to the editors. Each proposed problem or solution should be typed or printed neatly on separate sheets of paper with your name and affiliation (if desired) on each page. Include a self-addressed, stamped envelope if you would like the editors to acknowledge that they have received your contribution. Mail all contributions to Roger B. Nelsen. Submissions may be sent via e-mail to cmj@lclark.edu.

Solutions to the problems in this issue should be postmarked no later than April 15, 1993.

PROPOSALS

491. *Proposed by Michael Handelsman, Erasmus Hall High School, Brooklyn, NY.*

Let R be a rational function given by $R(x) = P(x)/x^n$, where P is a polynomial with real coefficients and $P(0) \neq 0$. Find $P(x)$ if $R'(1/x) = -x^2R'(x)$ for all $x \neq 0$. [$R'(1/x)$ denotes the derivative of R evaluated at $1/x$.]

492. *Proposed by Norman J. Finizio and James T. Lewis, University of Rhode Island, Kingston.*

A circular arrangement of the integers $\{1, 2, \ldots, 2n\}$ is *balanced* if $a + b = a' + b'$ whenever a and b are adjacent and a' and b' are diametrically opposite a and b. How many balanced circular arrangements of $\{1, 2, \ldots, 2n\}$ are there? [As usual, rotations are not counted as different.]

493. *Proposed by K. R. S Sastry, Addis Ababa, Ethiopia.*

Let $ABCDE$ be a convex, affinely regular pentagon in which each side is parallel to a diagonal. Let P, Q, R, S, T be points on the sides CD, DE, EA, AB, BC, respectively, so that AP, BQ, CR, DS, ET concur at X. Describe the set of points

X for which

$$\frac{XP}{PA} + \frac{XQ}{QB} + \frac{XR}{RC} + \frac{XS}{SD} + \frac{XT}{TE}$$

is constant.

494. *Proposed by Frank Schmidt, Arlington, VA.*

For $A \subseteq \{0,1\}$, let $N(A,n)$ be the number of $n \times n$ matrices over $\mathbf{Z}/2\mathbf{Z}$ whose set of eigenvalues in $\mathbf{Z}/2\mathbf{Z}$ is precisely A. Find a formula for $N(A,n)$.

495. *Proposed by William R. Klinger, Taylor University, Upland, IN.*

Determine whether or not there is a cubic polynomial with three distinct real zeros a_1, a_2, a_3 and two critical numbers c_1, c_2 $(a_1 < c_1 < a_2 < c_2 < a_3)$ such that the segment from a_1 to a_2 is divided harmonically by c_1 and c_2. [If the *cross-ratio* $(c_1 - a_1)(c_2 - a_2)/(c_1 - a_2)(c_2 - a_1)$ of a_1, a_2, c_1, and c_2 is -1, then c_1 and c_2 are said to divide a_1 and a_2 *harmonically*.]

SOLUTIONS

The Area of a Lattice Quadrilateral

466. (Jan. 1992) *Proposed by Ginger Bolton, Swainsboro, GA.*
Let k be a fixed positive integer and let

$$A = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid 10x + y \text{ is a multiple of } k\}.$$

Show that, if Q is a quadrilateral with vertices lying in A and no other point in A lies on either the boundary or interior of Q, then the area of Q is k.

Solution by F. C. Rembis, Clifton, NJ.

Consider a more general situation: Let $k, n \in \mathbf{Z}$, $k > 0$,

$$A = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid nx + y \text{ is a multiple of } k\},$$

and let Π be a simple polygon with vertices lying in A. (A polygon is *simple* if its boundary is a simple closed curve.)

Let φ: $\mathbf{Z} \times \mathbf{Z} \to \mathbf{Z} \times \mathbf{Z}$ be the map given by $(x,y) \mapsto (x,(nx+y)/k)$, and set $A' = \varphi A = \{(x',y') \in \mathbf{Z} \times \mathbf{Z}\}$. Since φ is composed of an area preserving linear shear and a contraction along y, Area $\Pi = k$ Area Π', where $\Pi' = \varphi(\Pi)$.

Applying Pick's theorem (see H. M. S. Coxeter, *Introduction to Geometry*, 2nd ed., Wiley, New York, 1989, p. 209) to Π', we have Area $\Pi' = q + p/2 - 1$, where q is the number of lattice points of A' interior to Π' and p is the number of lattice points of A' on the boundary of Π'. For the case in point, $q = 0$ and $p = 4$, so Area $\Pi = k$ Area $\Pi' = k$.

Also solved by C. PATRICK COLLIER, U. of Wisconsin, Oshkosh; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Denmark; JOHN GRAHAM, Pennsylvania State U., Wilkes-Barre; HELEN M. MARSTON, Princeton, NJ; PHILIP OPPENHEIMER, South Norwalk, CT; JAMES M. SWENSON, Long Beach, CA; and PETER L. VACHUSKA, U. of Wisconsin Centers, Washington County.

$\cos A : \cos B : \cos C = 2:9:12 \Rightarrow a:b:c = 6:5:4$

467. (Jan. 1992) *Proposed by Stanley Rabinowitz, Westford, MA.*

The cosines of the angles of a triangle are in the ratio $2:9:12$. Find the ratio of the sides of the triangle.

Solution by Jack V. Wales, Jr., The Thacher School, Ojai, CA.

The sides of the triangle are in the ratio $6:5:4$. There is a k such that $\cos A = 2k$, $\cos B = 9k$, and $\cos C = 12k$, where A, B, and C are the angles of the triangle. We see immediately that $\sin A = \sqrt{1-4k^2}$, $\sin B = \sqrt{1-81k^2}$, and $\sin C = \sqrt{1-144k^2}$. From the fact that $\cos(A+B) = -\cos C$ we get

$$432k^3 + 229k^2 - 1 = 0. \qquad (*)$$

The solutions to $(*)$ are $k = \frac{1}{16}$, $k = (-8+\sqrt{37})/27$, and $k = (-8-\sqrt{37})/27$. The latter two, being negative, do not apply here. Therefore, $\sin A = (6\sqrt{7})/16$, $\sin B = (5\sqrt{7})/16$, and $\sin C = (4\sqrt{7})/16$. The law of sines gives the ratio we seek.

Also solved by EDWARD ABOUFADEL, Rutgers U.; NIRMAL DEVI AGGARWAL, Embry Riddle Aeronautical U.; JOHN ANDRAOS, U. of Ottawa, Canada; GÜNTER BACH, Leinfelden, Germany; BRIAN D. BEASLEY, Presbyterian C.; FRANCISCO BELLOT ROSADO and MARÍA ASCENSIÓN LÓPEZ CHAMORRO (jointly), Valladolid, Spain; NIRDOSH BHATNAGAR, Cupertino, CA; S. C. BHATNAGER, U. of Nevada, Las Vegas; LAWRENCE S. BRADEN, St. Paul's School, Concord, NH; MICHAEL BROZINSKY, Queensborough C.C.; CENTRE C. PROBLEM SOLVING GROUP (two solutions); MARGARET CIBES, U. of Hartford; PHIL CLARKE, Los Angeles Valley C.; JOHN M. COKER, Shawnee Mission, KS; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Denmark; RAGNAR DYBVIK, Tingvoll, Norway; DAVID EARNSHAW, Lambton C., Canada; FRANK ECCLES, Phillips Academy, Andover, MA; MILTON P. EISNER, Mount Vernon C.; THOMAS E. ELSNER, GMI Engineering & Management Institute, Flint, MI; CAROLYN N. FOSTER, U. of Southern Maine; DONALD C. FULLER, Gainesville C.; JOHN GRAHAM, Pennsylvania State U., Wilkes-Barre; HAROLD W. HAGER, Southeast Missouri State U.; GEOFFREY A. KANDALL, Hamden, CT; KEE-WAI LAU, Hong Kong; LYNNWOOD LOWE, Darien, CT; PETER MADDEN, Amity H.S., Woodbridge, CT; BEATRIZ MARGOLIS, Paris, France; HELEN M. MARSTON, Princeton, NJ; KIM McINTURFF, Santa Barbara, CA; JOHN J. McKINLEY, Warner Robins, GA; W. WESTON MEYER, General Motors Research & Environmental Staff, Warren, MI; JOHN MORLINO, Georgian Court C.; JOHN I. NASSAR, Muhlenberg C.; PHILIP OPPENHEIMER, South Norwalk, CT; PENDER J. PEDLER, Edith Cowan U., Australia; THE PRESTONSBURG C.C. PROBLEM SOLVERS GROUP (two solutions); BILLY D. READ, Lamar U.; A. J. ROQUES, Louisiana State U., Eunice; JAMES RUDAITIS, Schenectady County C.C.; KAZEM S. SADATI, Penn Valley C.C.; WILLIAM SEAMAN, Albright C.; ROBERT W. SHEETS, Southeast Missouri State U.; SHREVEPORT PROBLEM GROUP; DEL SMUFA, Philadelphia, PA; JOHN S. SUMNER, U. of Tampa; R. S. TIBERIO, Natick, MA; S. A. TRUITT, Jr., Middle Tennessee State U.; U. OF ARIZONA PROBLEM SOLVING LABORATORY; COLLEEN A. and PETER L. VACHUSKA (jointly), U. of Wisconsin Centers, Washington County; MICHAEL VOWE, Therwil, Switzerland; WILLIAM V. WEBB, Akron, OH; HENRY G. WETZLER, Lake Ridge Academy, North Ridgeville, OH; DOUG WILCOCK, Istanbul, Turkey; PAUL YIU, Florida Atlantic U.; ROBERT L. YOUNG (two solutions), Cape Cod C.C.; ZHANG JIE-YE and JU-LIANNE F. STILE (jointly) (students), Allegheny C.; and the proposer. Three incorrect solutions were received.

Editors' Note. Most solutions received were similar to the one above. Beasley observes that this problem appeared in the 1980 Georgia Mathematics League high school contest in the following form: In triangle ABC, $\sin A : \sin B : \sin C = 4:5:6$, while $\cos A : \cos B : \cos C = x : y : 2$. Find the ordered pair (x, y).

Bounds on $\int_1^\infty \frac{dx}{1+x^{n+1}}$

468. (Jan. 1992) *Proposed by Murray Klamkin and Andy Liu (jointly), University of Alberta, Edmonton, Canada.*

If

$$I_n = \int_1^\infty \frac{dx}{1+x^{n+1}}, \qquad n > 0,$$

show that

$$\frac{\log 2}{n} + \frac{1}{4n^2} > I_n > \frac{\log 2}{n},$$

where $\log x$ is the natural logarithm function.

Solution by Joe Howard, New Mexico Highlands University, Las Vegas, NM.

Let $u = x^{-1}$, so $du = -x^{-2}\,dx$. Then

$$I_n = \int_1^\infty \frac{dx}{1+x^{n+1}} = \int_0^1 \frac{u^{n-1}\,du}{1+u^{n+1}}.$$

Now for $u \in (0,1)$

$$\frac{u^{n-1}}{1+u^n} + u^{2n-1} - u^{2n} > \frac{u^{n-1}}{1+u^{n+1}} > \frac{u^{n-1}}{1+u^n},$$

which can be easily verified. Integrating from 0 to 1 gives

$$\frac{\log 2}{n} + \frac{1}{2n} - \frac{1}{2n+1} > I_n > \frac{\log 2}{n},$$

or

$$\frac{\log 2}{n} + \frac{1}{4n^2+2n} > I_n > \frac{\log 2}{n}.$$

Also solved by SHIV KUMAR AGGARWAL and NIRMAL DEVI AGGARWAL (jointly), Embry Riddle Aeronautical U.; GÜNTER BACH, Leinfelden, Germany; PAUL BRACKEN and R. C. CENISEE (jointly), U. of Waterloo, Canada; PHIL CLARKE, Los Angeles Valley C.; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Denmark; DAVID EARNSHAW, Lambton C., Canada; JIRO FUKUTA, Gifu-ken, Japan; JOHN GRAHAM, Pennsylvania State U., Wilkes-Barre; EUGENE A. HERMAN, Grinnell C.; HONGWEI CHEN, Christopher Newport C.; PARVIZ KAJEH-KHALILI, Christopher Newport C.; KEE-WAI LAU, Hong Kong; H. K. KRISHNAPRIYAN, Drake U.; ALBERT KURZ (student), Council Rock H.S., Newtown, PA; ANTHONY LO BELLO and TIAN YUAN JACKIE (student) (jointly), Allegheny C.; KIM McINTURFF, Santa Barbara, CA; PAUL MURRIN and A. J. ROQUES, Louisiana State U., Eunice; NORTHERN KENTUCKY U. PROBLEM GROUP; PHILIP OPPENHEIMER, South Norwalk, CT; PRESTONSBURG C.C. PROBLEM SOLVERS GROUP; F. C. REMBIS, Clifton, NJ; WILLIAM SEAMAN, Albright C.; H.-J. SEIFFERT and K. HOFFEINS (jointly), Berlin, Germany; JOHN S. SUMNER, U. of Tampa; S. A. TRUITT, Jr., Middle Tennessee State U.; COLLEEN A. and PETER L. VACHUSKA (jointly), U. of Wisconsin Centers, Washington County; MICHAEL VOWE, Therwil, Switzerland; JOSEPH WIENER, U. of Texas-Pan American; ZHANG ZAIMING, Yuxu Teachers' C., China; and the proposers.

Editors' Note. Many solvers improved the upper bound to $\log(2)/n + 1/(4n^2+2n)$. Chen and Seiffert (independently) improved the lower bound to $\log(2)/n + (n+2)/6n(2n+1)(3n+2)$. Seiffert also gave the following generalization:

If

$$I_{m,n} = \int_1^\infty \frac{x^m}{1+x^{n+1}}\,dx$$

with $-1 < m < n$, then

$$\frac{\log(2)}{n-m} + \frac{m+1}{2(n-m)(2n-m+1)} > I_{m,n} > \frac{\log(2)}{n-m} + \frac{(m+1)(n+m+2)}{6(n-m)(2n-m-1)(3n-m+2)}.$$

A Sum of 49 Squares

469. (Jan. 1992) *Proposed by K. R. S. Sastry, Addis Ababa, Ethiopia.*

Characterize the natural number arithmetic progressions of length 49 such that the sum of the squares of the terms are again integral squares.

Solution by Michael Vowe, Therwil, Switzerland.

The trivial case is if all 49 numbers are equal.

Denoting the 49 natural numbers by $a_1, a_2, \dots, a_{49}$, the common difference by $d > 0$, and the middle number a_{25} by x, we obtain

$$\sum_{i=1}^{49} a_i^2 = x^2 + \sum_{i=1}^{24} \left[(x - id)^2 + (x + id)^2\right]$$
$$= 49x^2 + 2d^2 24 \cdot 25 \cdot 49/6 = 49(x^2 + 200d^2).$$

Putting $10d = y$ we have to look for the solutions of the Diophantine equation

$$x^2 + 2y^2 = z^2 \qquad (1)$$

where $y/10$ is a natural number and $x > 2.4y$ (since $a_1 = x - 24d > 0$).

The solutions of (1) are [cf. L. E. Dickson, *History of the Theory of Numbers*, Vol. II, Chelsea, New York, 1971, p. 426]

$$x = r(s^2 - 2t^2), \qquad y = 2rst$$

where s, t are relatively prime, $rst/5$ a natural number and $s \geq 6t$, (since $x > 2.4y \Leftrightarrow s^2 - 2t^2 > 4.8st \Leftrightarrow (s - 2.4t)^2 > 7.76t^2$).
Examples:

(1) $r = 1, t = 1, s = 10$, i.e. $x = 98, d = 2, 50^2 + 52^2 + \cdots + 146^2 = (7 \cdot 102)^2$.

(2) $r = 5, t = 1, s = 6$, i.e. $x = 170, d = 6, 26^2 + 32^2 + \cdots + 314^2 = (7 \cdot 190)^2$.

Also solved by MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I. B. "Leopoldo Cano," Spain; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Denmark; RAGNAR DYBVIK, Tingvoll, Norway; JIRO FUKUTA, Gifu-ken, Japan; RICKY IKEDA, Leeward C. C.; H. K. KRISHNAPRIYAN, Drake U.; PRESTONSBURG C.C. PROBLEM SOLVERS GROUP; F.C. REMBIS, Clifton, NJ; H.-J. SEIFFERT and K. HOFFEINS (jointly), Berlin, Germany; JOHN S. SUMNER and KEVIN L. DOVE (jointly), U. of Tampa; and the proposer. There were four incomplete solutions.

$\lim \frac{1}{n \sin n}$ Does Not Exist

470. (Jan. 1992) *Proposed by William Wardlaw, U.S. Naval Academy, Annapolis, MD.*

Evaluate

$$\lim_{n \to \infty} \frac{1}{n \sin n}$$

or show that it does not exist.

Solution by Eugene A. Herman, Grinnell College, Grinnell, IA.

The limit does not exist. We use the following well-known theorem on rational approximations [see Ivan M. Niven and Herbert S. Zuckerman, *An Introduction to the Theory of Numbers*, Wiley, New York, 1960, p. 132]: For every irrational x,

$$\left|\frac{n}{k}-x\right| \le \frac{1}{k^2} \quad \text{for infinitely many integers } n \text{ and } k. \tag{1}$$

Applying (1) with $x=\pi$, we have, for infinitely many n and k,

$$|n-k\pi| \le \frac{1}{k} \quad \text{and} \quad \frac{n}{k} \le \pi + \frac{1}{k^2} \le \pi + 1.$$

Using also $|\sin\theta| \le |\theta|$, we conclude that

$$\left|\frac{1}{n\sin n}\right| = \frac{1}{n|\sin(n-k\pi)|} \ge \frac{1}{n|n-k\pi|} \ge \frac{1}{n/k} \ge \frac{1}{\pi+1}. \tag{2}$$

Next we note that $|\sin\theta| - |\sin\varphi| \le |\sin\theta - \sin\varphi| \le |\theta-\varphi|$, or $|\sin\varphi| \ge |\sin\theta| - |\theta-\varphi|$. Thus, for the same values of n and k as above, with $\varphi = n+1-k\pi$ and $\theta = 1$, we have $|\sin(n+1-k\pi)| \ge |\sin 1| - |n-k\pi|$. Finally,

$$\begin{aligned}|(n+1)\sin(n+1)| &= (n+1)|\sin[(n+1)-k\pi]| \\ &\ge (n+1)(|\sin 1| - |n-k\pi|) \\ &\ge (n+1)\left(|\sin 1| - \frac{1}{k}\right) \to \infty. \end{aligned} \tag{3}$$

Conclusions (2) and (3) together imply that $\lim_{n\to\infty} \frac{1}{n\sin n}$ does not exist, since both hold for infinitely many n.

Also solved by GÜNTER BACH, Leinfelden, Germany; LARRY BLAINE, Plymouth State C.; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Denmark; JOHN GRAHAM, Pennsylvania State U., Wilkes-Barre; DON JONES, KATHLEEN SHANNON, and HOMER AUSTIN (jointly), Salisbury State U.; H. K. KRISHNAPRIYAN, Drake U.; RICK MABRY and KEITH NEU (student) (jointly), Louisiana State U.; NORTHERN KENTUCKY U. PROBLEM GROUP; JOHN S. SUMNER and KEVIN L. DOVE (jointly), U. of Tampa; and the proposer. There was one incomplete solution and four incorrect solutions.