techniques explored in general in his paper, "Power identities for sequences defined by $W_{n+2} = dW_{n+1} - cW_n$", *Fibonacci Quarterly*, 3 (1965), 241–256. Francisco Bellot and María Ascensión López pointed out that the particular case with $a = b = d = 1$ and $c = 4$ was a problem from the 1987 Bulgarian Mathematical Olympiad. A solution and generalization to that problem appeared in *Crux Mathematicorum*, 16 (1990), 292–294.

Solved by 49 other readers and the proposer.

Nearest Integer Zeta Functions

10212 [1992, 361]. *Proposed by Seung-Jin Bang, Seoul, Korea.*

Let $a(n)$ be the integer closest to $\sqrt[3]{n}$. Evaluate $\sum_{n=1}^{\infty} a(n)^{-4}$.

Composite solution by all solvers. The value is $\pi^2/2 + \pi^4/23040$. Indeed, for $s > 3$, one has

$$\sum_{n=1}^{\infty} a(n)^{-s} = 3\zeta(s-2) + 4^{-s}\zeta(s).$$

Here, $\zeta(s)$ denote the Riemann zeta function. The values of $\zeta(2n)$ for positive integers are easily computed rational multiples of π^{2n}.

Proof: The expression to be evaluated may be written as $\sum_{r=1}^{\infty} f(r) r^{-s}$ where $f(r)$ is the number of positive integers n for which $\sqrt[3]{n}$ is closest to r. Thus $f(r)$ is the number of positive integer solutions of

$$\left(r - \frac{1}{2}\right)^3 < n < \left(r + \frac{1}{2}\right)^3.$$

(The endpoints may be excluded since they cannot be integers.) This is an interval of length $3r^2 + 1/4$, so it will contain either $3r^2$ or $3r^2 + 1$ integers (all positive if $r > 0$). The number is $3r^2 + 1$ only if $(2r+1)^3 \equiv 1 \pmod 8$, and this is true if and only if $r \equiv 0 \pmod 4$. Thus

$$\sum_{n=1}^{\infty} a(n)^{-s} = \sum_{r=1}^{\infty} f(r) r^{-s} = \sum_{r=1}^{\infty} (3r^2) r^{-s} + \sum_{m=1}^{\infty} (4m)^{-s}$$
$$= 3\zeta(s-2) + 4^{-s}\zeta(s).$$

Editorial comment. Jonathan M. Borwein & Leo C. Hsu, Rick Mabry & Keith Neu, Josef Roppert, Douglas B. Tyler, and B. M. M. de Weger considered the more general sums $S_N(s) = \sum_{n=1}^{\infty} a_N(n)^{-s}$ where $a_N(n)$ is the integer closest to $\sqrt[N]{n}$. The method employed above for $N = 3$ also gives $S_2(s) = 2\zeta(s-1)$ and $S_4(s) = 4\zeta(s-3) + \zeta(s-1)$. For larger N, Hurwitz zeta functions, $\zeta(a, s) = \sum_{n=0}^{\infty}(n + a)^{-s}$ appear, and only Chu and the team of Borwein and Hsu persisted to give any values of $S_N(n)$ with $N > 4$. The latter solution contains a proof that $S_N(n)$ is a polynomial in π whose coefficients are algebraic numbers whenever $n - N$ is an odd integer. For example, with

$$Q_6 = \frac{170912 + 49928\sqrt{2}}{15} \quad \text{and} \quad Q_7 = \frac{246013 + 353664\sqrt{2}}{45},$$

one gets

$$S_5(6) = \frac{5\pi^2}{6} + \frac{\pi^4}{36} + \left(\frac{1}{945} - Q_6\sqrt{1 - \sqrt{\frac{1}{2}}}\right)\frac{\pi^6}{4^{12}}$$

$$S_6(7) = \pi^2 + \frac{\pi^4}{18} + \frac{\pi^6}{2520} + Q_7\frac{\pi^7}{2^{27}}.$$

Solved by 65 readers (including those cited) and the proposer. One incorrect solution was received.

A Ring with No Nilpotents

10215 [1992, 362]. *Proposed by Michael Barr, McGill University, Montreal, Quebec, Canada.*

Let R be an associative ring (not necessarily commutative or possessing a unit element) with no non-zero nilpotent elements. Suppose that r and s are two elements of R such that $r^d = s^d$ and $r^e = s^e$, where d and e are relatively prime positive integers. Show that $r = s$.

Solution by Pat Stewart, Dalhousie University, Halifax, Nova Scotia, Canada. Choose integers a and b such that $ad + be = 1$. Without loss of generality, we may assume that $a > 0$ and $b < 0$. Since $r^d = s^d$, we have $r^{ad} = s^{ad}$ and hence $r^{1-be} = s^{1-be}$. Also, since $r^e = s^e$, we have $r^{-be} = s^{-be}$. Thus there is a positive integer u such that $r^{1+u} = s^{1+u}$ and $r^u = s^u$.

Substituting for s^u, we see that $r^{1+u} = r^u s$, and hence $r^u(r - s) = 0$. Elements x, y of a ring with no non-zero nilpotent elements satisfy $xy = 0$ if and only if $yx = 0$. Hence $r(r - s)r^{u-1} = 0$. Continuing in this fashion, one obtains by induction that $(r(r - s))^u = 0$, and so $r(r - s) = 0$. Similarly, $s(r - s) = 0$. Hence $(r - s)^2 = 0$, which implies $r - s = 0$, as desired.

Editorial comment. Several readers noted that the term *reduced* is used for rings without nilpotent elements. D. D. Anderson sketched an argument to show that, in a reduced ring, if $a_1 \dots a_n = 0$ and σ is any permutation of $[1, \dots, n]$, then $a_{\sigma(1)} \dots a_{\sigma(n)} = 0$. Frank Schmidt noted that the problem follows from the *structure theorem* for reduced rings: every reduced ring is a subdirect product of domains. W. K. Nicholson made the same observation, including a reference to source of the theorem (Andrunakevič and Rjabuhin, Soviet Math. Doklady 9 (1968), 565–567, MR 37 #6320) and the proof by A. Klein (Canad. Math. Bull. 23 (1980), 495–496).

Solved also by D. D. Anderson, G. Behrendt (Germany), B. W. Brock, D. Caccia, D. Callan, R. J. Chapman (U. K.), T. C. Craven, E. Dobrowolski and N. Buck (Canada), N. J. Fine, E. A. Herman, M. Hongan (Japan), M. Juvan (Slovenia), S. Kanetkar, K. S. Kedlaya (student), J. F. Kennison, J. J. Kuzmanovich, C. Lanski, S. C. Locke, O. P. Lossers (The Netherlands), R. F. McCoart, Jr., A. Mülller (France), W. K. Nicholson (Canada), F. Schmidt, R. Stong, E. T. Wong, University of Wyoming Problem Circle, and the proposer.