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Some remarks concerning the uniformly gray sets of G. Jacopini(**)

ABSTRACT. — In a 1995 paper by G. Jacopini, a σ-algebra $\mathcal{H}$ of subsets of $\mathbb{R}$ is constructed, and a translation-invariant measure ν extending the Lebesgue measure λ is defined on $\mathcal{H}$, such that for each $r \in [0,1]$ there are $E \in \mathcal{H}$ for which $\nu(E \cap A) = r\,\lambda(A)$ for all Borel subsets A. Given these properties, it can be suggested, as an intuitive interpretation, that such sets as E are ≪uniformly gray≫. The main purpose of this note is to discuss the extent to which such an intuitive characterization is reasonable for sets having the above properties. Also mentioned are some other results, similar to Jacopini's, which have been published elsewhere.

Alcune osservazioni sugli insiemi uniformemente grigi di G. Jacopini

RIASSUNTO. — In un articolo del 1995, G. Jacopini esibisce una σ-algebra $\mathcal{H}$ di parti di $\mathbb{R}$ e una misura ν su di essa definita, invariante per le traslazioni, prolungante la misura di Lebesgue λ e tale che, per ogni numero reale r compreso tra 0 e 1, esista un elemento E di $\mathcal{H}$ verificante la relazione $\nu(E \cap A) = r\,\lambda(A)$ per ogni insieme boreliano A. Questa proprietà di E può suggerire, dal punto di vista intuitivo, l'idea di un insieme ≪uniformemente grigio≫. Lo scopo principale della presente nota consiste nel discutere fino a qual punto una simile interpretazione intuitiva sia ragionevole, anche alla luce di risultati analoghi a quello di Jacopini, pubblicati da altri autori.

In the 1995 paper [4] of G. Jacopini, the following result, which we shall call THEOREM J, is stated and proved.

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THEOREM: There exists a σ-algebra $\mathcal{H}$ of $\mathbb{R}$, properly containing the Borel σ-algebra, and a measure ν on $\mathcal{H}$ extending the Lebesgue measure λ and having the two following properties:
(a) ν is translation-invariant, in the sense that for each element A of $\mathcal{H}$ and each real number t, one has

$$A + t \in \mathcal{H} \quad \text{and} \quad \nu(A + t) = \nu(A).$$

(b) For each real number r between 0 and 1, there is an element E of $\mathcal{H}$ such that for every Borel set A of $\mathbb{R}$, one has

$$\nu(E \cap A) = r\, \lambda(A).$$

Professor Jacopini offers the intuitive interpretation: "se si immagina di colorare in nero i punti di E, e in bianco quelli del complementare,"[1] then such sets as E can be thought of as being "≪uniformemente grigio≫"[2].

In the 1991 paper [14], the author of the present note gives various constructions, some not too dissimilar from the above (see, e.g., Example 4.7 of that paper), of subsets of $\mathbb{R}$ called *shadings*, and whose definition (below) also suggests ≪shades of gray≫. (In case the notion of color or shade on a line is not intuitively appealing, we mention that similar constructions can be given for subsets of the plane.)

Even though it was not explicitly stated in [4], we naturally would hope to associate the number r mentioned in the theorem with the ≪darkness≫, ≪gray-scale≫ or ≪shade of gray≫, the shades increasing continuously from white ($r = 0$) to black ($r = 1$).

What we shall discuss now is the extent to which it may or may not be reasonable to consider such sets as those with properties (a) and (b) described in THEOREM J as actually having a *definite* shade of gray in any sense whatsoever. In fact, to the contrary, we note the existence of σ-algebras of subsets of $\mathbb{R}$ equipped with translation-invariant measures, each of which share the properties (a) and (b) in THEOREM J, but for which it is clearly *not* reasonable to think of the number r as having anything whatsoever to do with an increasing gray-scale as described above. A notable case is found in the famous 1950 paper [5] of Kakutani and Oxtoby, in which a nonseparable extension of the Lebesgue measure is constructed, which is isometry-invariant and whose character is maximal ($2^{\mathfrak{c}}$). (We will use the symbol $\mathfrak{c}$ to refer to the cardinality

[1] English translation: "if one imagines coloring the points of E with black and the points of the complement with white"

[2] "≪uniformly gray≫"

of the continuum, and also, when convenient, the smallest ordinal having this cardinality.) To make our point, we need only observe the following, which is a consequence of the construction in [5].

PROPOSITION: *There exists a σ-algebra $\mathcal{K}$ of $\mathbb{R}$, properly containing the Borel σ-algebra, and a family of measures $\{\nu_\alpha\}_{\alpha<\mathfrak{c}}$, each ν_α defined on $\mathcal{K}$ and extending the Lebesgue measure λ, such that the two following properties hold:*

(a) Each ν_α is translation-invariant, in the sense that for each element A of $\mathcal{K}$ and each real number t, one has

$$A+t\in\mathcal{K} \quad \text{and} \quad \nu_\alpha(A+t)=\nu_\alpha(A).$$

(b) For every transfinite sequence $(r_\alpha)_{\alpha<\mathfrak{c}}$ whose members are contained in the unit interval $[0,1]$, there is an element C of $\mathcal{K}$ such that for every Borel set A of $\mathbb{R}$, one has (for the same C)

$$\nu_\alpha(C\cap A)=r_\alpha\,\lambda(A) \quad \forall\alpha<\mathfrak{c}.$$

We emphasize that the σ-algebra $\mathcal{K}$ and the family of measures $\{\nu_\alpha\}_{\alpha<c}$ are fixed; C depends only upon the sequence $(r_\alpha)_{\alpha<\mathfrak{c}}$. In particular, *all of the r_α may be chosen to be distinct.* So what we have is a family of translation-invariant extensions of λ, each measuring C differently. We sketch the proof of this proposition below.

In what follows, the cardinality of a set X is denoted by $|X|$. A subset X of $\mathbb{R}$ will be called *almost invariant* with respect to a group G of transformations on $\mathbb{R}$ provided that $|X\bigtriangleup g(X)|<\mathfrak{c}$, for all $g\in G$.

SKETCH OF PROOF OF PROPOSITION: The necessary fact is that there exists a family $\{C_\alpha\}_{\alpha<\mathfrak{c}}$ of subsets of $\mathbb{R}$ such that:

(1) $\{C_\alpha\}_{\alpha<\mathfrak{c}}$ is a partition of $\mathbb{R}$, i.e.,

$$\mathbb{R}=\biguplus_{\alpha<\mathfrak{c}}C_\alpha.$$

(We use $\biguplus$ to emphasize *disjoint* unions.)

(2) Each C_α is almost invariant relative to the group of all isometries on $\mathbb{R}$.

(3) If A is a Borel subset of $\mathbb{R}$ and $\lambda(A)>0$, then $C_\alpha\cap A\neq\emptyset$ for each $\alpha<\mathfrak{c}$.

These properties imply that for each $\alpha < \mathfrak{c}$, the set C_α has zero inner measure and full outer measure. (I.e., $\lambda_*(C_\alpha) = 0$ and $\lambda^*(C_\alpha \cap A) = \lambda(A)$ for each Borel set A. Thus, C_α is *saturated nonmeasurable*). We let $\mathcal{L}$ denote the Lebesgue measurable sets, $\mathcal{C}$ the family $\{C_\alpha\}_{\alpha<c}$, and $\mathcal{N}$ the noncontinuum subsets of $\mathbb{R}$ (i.e., $X \in \mathcal{N}$ iff $|X| < \mathfrak{c}$). Then the σ-algebra $\mathcal{K}$ generated by $\mathcal{C} \cup \mathcal{L}$ is translation-invariant and contains elements of $\mathcal{N}$.

Since $\aleph_0 \cdot \mathfrak{c} = \mathfrak{c}$, we may subdivide the partition $\mathcal{C}$ as follows:

$$\mathcal{C} = \biguplus_{i=1}^{\infty} \biguplus_{\alpha<c} C_\alpha^{(i)}.$$

With these properties, if $(x_1, x_2, x_3, \dots)$ is any sequence whatsoever of nonnegative numbers for which $\sum_{i=1}^{\infty} x_i = 1$, then measures $\{\nu_\alpha\}_{\alpha<\mathfrak{c}}$ may be defined on $\mathcal{K}$ by means of the assignments

$$\nu_\alpha(X) = 0, \quad \forall X \in \mathcal{N} \cap \mathcal{K},$$

and

$$\nu_\alpha(C_\beta^{(i)} \cap A) = \begin{cases} x_i \; \lambda(A) & \text{if } \beta = \alpha \\ 0 & \text{if } \beta \neq \alpha \end{cases} \qquad \forall A \in \mathcal{L}, \; i = 1, 2, 3, \dots .$$

Specifically, we shall take

$$x_i = 1/2^i, \quad \forall i.$$

For each $x \in [0, 1]$, let

$$x = 0.b_1(x)\; b_2(x)\; b_3(x) \cdots \quad \text{(base 2)}$$

be the usual dyadic (nonterminating) expansion of the number x, and let

$$N(x) = \{i : b_i(x) = 1\}.$$

Then

$$\sum_{i \in N(x)} x_i = x,$$

and so, letting

$$C = \bigcup_{\alpha<\mathfrak{c}} \bigcup \{C_\alpha^{(i)} : i \in N(r_\alpha)\},$$

it is easy to verify that the measures $\{\nu_\alpha\}_{\alpha<\mathfrak{c}}$ and the set C will have the desired properties. $\square$

Thus, the set C may be considered to have an arbitrarily assigned (although uniform) shade of gray, with respect to such measures, including completely black ($r = 1$) or white ($r = 0$). Given this ambiguous situation, it is not reasonable to associate with the set C any definite shade whatsoever.[3]

In connection with this, we should mention that in [6], A.B. Kharazishvili observes this phenomenon of ambiguity, referring also to the construction in [5], by noting that "the uniqueness property [relative to the class of isometry-invariant extensions of λ] so characteristic for Lebesgue measure is violated here." Briefly, for our purposes, a subset X of $\mathbb{R}$ has the *uniqueness property* in a class $\mathfrak{M}$ of extensions of λ provided that X is μ-measurable for some $\mu \in \mathfrak{M}$ and that whenever X is μ'-measurable for some $\mu' \in \mathfrak{M}$, then $\mu(X) = \mu'(X)$. See, for example, the extensive monograph [9] (§7) or the paper [8].

In the same paper [6], Kharazishvili then proceeds to construct a nonseparable isometry-invariant extension of λ (having character $\mathfrak{c}$) which does indeed have the uniqueness property in the class of all isometry-invariant extensions of λ. In fact, while not explicitly stating it in [6], the measure constructed therein shares the very same properties of the measure ν as set forth in the statement of Theorem J. Having the uniqueness property in addition to the aforementioned properties, the sets so obtained might be said to have specific, definite shades of gray.

For comparison, let us now suggest another approach to describing shades of gray. Recall that a Banach measure on $\mathbb{R}$ (resp., $\mathbb{R}^2$) is an isometry-invariant extension of the Lebesgue measure λ defined on all subsets of $\mathbb{R}$ (resp., $\mathbb{R}^2$). Such a measure, whose existence is guaranteed by the Hahn-Banach theorem, is only finitely-additive. (See the excellent book [15] of Stan Wagon for many details concerning Banach measures.)

Definition: (cf. [14]) Let $\mathfrak{B}$ denote the class of all Banach measures on $\mathbb{R}$. If D is a subset of $\mathbb{R}$ for which there is a number $r \in [0, 1]$ such that

$$\mu(D \cap A) = r\, \lambda(A)$$

for all bounded Borel sets A and all $\mu \in \mathfrak{B}$, then we call r the *shade* of D, and we write $\mathrm{sh}(D) = r$. We call D an r-shading of $\mathbb{R}$.

[3] One wonders if it is even reasonable to ask, "What does such a set really look like?" for a set such as C, or indeed, for any of the sets mentioned in this note.

Simply put, this definition requires that such a shading of $\mathbb{R}$ has uniform[4] density and the uniqueness property in the class $\mathfrak{B}$.

In spite of the restrictions associated with finite-additivity, there are advantages in defining the shade of a set as above, rather than, say, requiring that sets have uniform density and the uniqueness property in the class of all *countably-additive* isometry-invariant extensions of λ. For one thing, the disadvantage of finite-addivity is greatly offset by the advantage of being able to measure all sets! (For interesting discussions of the relative merits of various extensions of Lebesgue measure, see the articles [1] and [2].) But moreover, consider the following subset W of $\mathbb{R}$, which has shade equal to zero[5] ([14], Example 4.2). Let H be a Hamel basis for $\mathbb{R}$ (over $\mathbb{Q}$) and fix any single element $h \in H$. Define W to be the set of all $x \in \mathbb{R}$ for which the rational coefficient of h is zero in the representation of x with respect to H. Then $\mathbb{R}$ is partitioned into countably many disjoint translates of W:

$$\mathbb{R} = \biguplus_{q \in \mathbb{Q}} (W + qh).$$

As we mentioned, $\mathrm{sh}(W) = 0$, which is certainly intuitively appealing in view of the observed partition. But the shade cannot be described in terms of countably-additive extensions of λ, for the set W is not even ν-measurable for *any* countably-additive translation-invariant extension of λ. (I.e, W is *absolutely nonmeasurable* in this class of measures. See, e.g., [9, §4], [12, §2], or [7].) To see this, observe that if $\nu(W \cap [0, 1)) = 0$, then one obtains the contradiction $\nu(\mathbb{R}) = 0$, because the union of countably many disjoint translates of $W \cap [0, 1)$ can equal all of $\mathbb{R}$. On the other hand, if $\nu(W \cap [0, 1)) > 0$, then one obtains the contradiction $\nu([0, 1)) = \infty$, by observing that for every $\varepsilon > 0$, infinitely many disjoint translates of $W \cap [0, 1)$ exist in $[0, 1 + \varepsilon)$.

It must also be noted that in [12, §8], a construction is given which has a great deal in common with the construction in [4]. (The construction may also

[4] We should also add that the notion of shade can be generalized as follows. If $f : \mathbb{R} \to [0, 1]$ is a continuous function, then there exists a subset D of $\mathbb{R}$ for which the *shade at each point* $x \in \mathbb{R}$, denoted by $\mathrm{sh}(D)(x)$, exists and is equal to $f(x)$, where $\mathrm{sh}(D)(x)$ is defined by $\lim_{\delta \to 0+} \mu(D \cap (x - \delta, x + \delta))/(2\delta)$, the values of $\mu(D \cap (x - \delta, x + \delta))$ being independent of $\mu \in \mathfrak{B}$. This is proved in [14]. Thus, sets can be *smoothly shaded.* (See [10] for an analogous result involving the uniqueness property for countably-additive invariant extensions of λ.)

[5] The set W is an example of a *nontrivial homogeneous set* in the sense of [3]: W is uncountable, not equal to $\mathbb{R}$, and $W + w_1 - w_2 = W$ for all $w_1, w_2 \in W$. In that paper it is proved that every nontrivial homogeneous set has shade equal to zero (although it is certainly not stated in such terms).

be found in [11]. Also, see [10] and the famous paper [13] for invariant extensions of λ using many of the same ideas.)

In particular, on p. 148 of [12], the σ-algebra S of [12] is analagous to the σ-algebra $\mathcal{H}$ of [4]. In fact, using the notation of [12], if one lets $0 \leq r \leq 1$ and $F = \{x \in \mathbb{R} : (x, f(x)) \in \mathbb{R} \times [0, 2\pi r)\}$, then the set F satisfies the properties possessed by E as stated in Theorem J with the additional property that F is a Bernstein set.

Our final remark (which perhaps should have been made earlier!) is that, despite our having gone to such lengths to point out the possible ambiguity of the shades of sets having the properties stated in Theorem J, it is not difficult to show that the particular constructions given for E in [4] and F (as above) in [12] yield sets which *do indeed* have the uniqueness property (in the class of all translation-invariant extensions of λ).

In fact, the measures on the σ-algebras S in [12] and $\mathcal{H}$ in [4] are *uniquely defined* in the class of all translation-invariant extensions of λ (cf. [9]), which means that all of the elements in S and $\mathcal{H}$ have the uniqueness property. (The same can be said for the measure constructed in [13], as noted in [10].) The proof of this uniqueness relies upon the essence of the final Remark given in [4], which we shall put in the following terms: the measure ν has the *property of exhaustion* with respect to translations in $\mathbb{R}$. This means that for $X \in \mathcal{H}$ with $\nu(X) > 0$, there exist $(t_i)_{i\in\mathbb{N}}$ for which

$$\nu\left(\mathbb{R} \setminus \bigcup_{i\in\mathbb{N}} (X + t_i)\right) = 0.$$

(That is, ν is *metrically transitive* with respect to X.) And in the class of translation-invariant extensions of λ, the uniqueness property and the exhaustion property are equivalent (see [6], Proposition 1).

Just one last thing: the sets E and F are r-shadings of $\mathbb{R}$. Furthermore, this author knows of no subset of $\mathbb{R}$ having uniform density and the uniqueness property (relative to all isometry-invariant countably-additive extensions of λ) for which X is not also a shading.

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