Supplement to "Branko's M2 is Brunnian"

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Contents

This discussion is the promised "Supplement" to the note [0], where most of the needed terminology is presented.

1 Only M2.

The Brunnian link "AltM2" in [0] is an *alternating* version of the simple monotone symmetric Venn diagram called "M2" in [11, sec: Examples of Symmetric Diagrams for small n]. As described in [0], we are considering simple Venn diagrams as minimal projections of alternating links. Conversely, we can take a simple Venn diagram and "lift it to" or "weave it into" an alternating link and that is what we shall mean when we refer to "the alternating link of a Venn diagram" or similar phrasings.

The Venn diagram M2 was first discovered by Branko Grünbaum [4, Figure 6] and was the first known simple symmetric Venn diagram on 7 Venn sets. We can state here that M2, quite remarkably, turns out to be the *only* one of the 56 known simple symmetric Venn diagrams on 7 sets whose alternating link is Brunnian. We used custom *Mathematica* programs to assist in this determination, some of the methods and results of which are described in the next sections. All 56 examples can now be checked by hand using the data we provide here and in conjunction with figures and encodings available on the extensive online dynamical survey [11]. It is an arduous task if done from scratch, but using our results things can be verified relatively easily.

The curves of every simple diagram, Venn or not, form an *alternating link diagram* when each of those curves are "over-under-ized" by weaving at the diagram's *crossings* as one travels around the curve. The result of doing this to a simple Venn diagram of order n produces what we call here an *alternating Venn link*. If the Venn diagram is (rotationally) *symmetric* then the result is an n-component link that is homeomorphic to one that is invariant in $\mathbb{R}^3$ under rotations by $2\pi/n$ about an (obvious) axis. We'll call such a thing a *symmetric* alternating Venn link. (Knot theorists usually call such links *n-periodic.*)

We may also consider symmetric *non*-alternating links that project to symmetric Venn diagrams. Every simple Venn diagram of order n has $2^n - 2$ crossings, so 126 crossings for $n = 7$. Cutting a symmetric 7-Venn into identical sevenths[1] generates $126/7 = 18$ crossings to consider in the link diagram, where we get 2^{18} different ways to assign either "over" or "under" at each crossing. For each such choice, the given crossings can be applied to each seventh in exactly the same way and a symmetric link results. Given one of these assignments we can reverse all of its crossings to obtain a new link, but that one will be a mirror image of the other and we won't consider that to be different. (That is, we don't care here about differences in *chirality.* If a link is Brunnian, then so is its mirror image, in a trivial way.) Thus for each symmetric Venn diagram of order 7 (such as M2), we can check $2^{17} = 131\,072$ distinct weavings that produce the entire collection of what we'll call the diagram's *symmetric Venn links.* These can be tested one by one for cases of Brunnianism.

[Let us interrupt this narrative to ask, Who cares? Well, there is at least precedent in the article [7], where Brunnian links associated with certain Venn diagrams are examined. That can suffice as an excuse for our proceedings. If another motivation is needed, let us acknowledge that Venn diagrams are interesting in their own right, as are Brunnian links. So it makes sense to ask when these two interesting objects can be so closely interwined!]

After an exhaustive search for symmetric 7-Venn links, we can report (or perhaps we should just say, "claim") that among all the known symmetric 7-Venns, the only symmetric Brunnian example is the alternating case of M2. This seems remarkable.

There likewise are no symmetric Brunnian 5-Venns other than the alternating one in Figure 8 (also shown in Figure 4 of [0]). For $n = 3$ and $n = 2$, the alternating forms give the only Brunnian cases, in the form of the Borromean rings and the Hopf link, respectively (the adjective "symmetric" being redundant in the cases of $n = 3$ and $n = 2$, as there is just one simple Venn diagram for each (see [11])).

Question 1 *So what's up with that? If a symmetric Brunnian link exists, must it be alternating?*

Perhaps when the hundreds of thousands of newly discovered ([8, 9]) simple symmetric 11-Venns are examined (along with the possibly billions of 13-Venns) more cases will turn up. Or maybe there will be no Brunnian examples at all.

[1] Fermat's little theorem implies that $n|2^n - 2$ when n is prime.

(A high-stakes beer bet exists between the first author of this article and the second author of the two cited articles.) A case for pessimism might be this: It seems reasonable to suppose that when the conditions of alternation and symmetry are dropped, it should *then* at least always be possible to find, for each Venn diagram, some weaving—some set of crossings—that would produce a Brunnian link. But that turns out not to be true, as noted in the final remark of [7]. A case for optimism could be, however, that the number of symmetric diagrams grows explosively for higher n (which must be prime for ths symmetric cases) and the number of variations in crossings for each such diagram also grows double-exponentially.

2 Polar, monotone, and non-monotone Venn diagrams.

We refer the reader to [11] for definitions of *monotone* Venn diagrams and symmetric Venn diagrams having *polar symmetry*, though neither of these is really necessary for our discussion. For our purposes the differences mainly have to do with how the different diagrams are named and listed. Ultimately, though, monotonicity does play a role here. First because while the precise number of monotone cases of simple symmetric 7-Venns is known, the same cannot yet be said with complete certainty for the non-monotone ones. (It is simply easier to crank out all the monotone cases, which we have done independently, confirming the list given in [11].) Secondy, non-monotonicity might actually increase the difficulty in determining Brunnianism in the resulting links. This is hinted at in the last remark in [7], where results about braid groups are employed. We use far cruder methods than that, so for us, monotonicity plays no role. We need only an encoding of the diagram.

3 About the method

For now we'll consider only the alternating cases of the symmetric 7-Venn links. Later we briefly mention what happens with the non-alternating versions.

Our initial method for seaching the simple symmetric 7-Venns for Brunnanism was admittedly somewhat crude and naive. We're looking for cases where every 6-component sublink of the 7 components is unlinked, so the first thing we can do (since it is easiest) is to eliminate any cases where there are two linked components. From the survivors of this test we eliminate any for which some three components are linked, and so on. Notice that when we proceed in this way, when we come to a "failure"—a sublink with k coponents that is not unlinked—we automatically have a Brunnian link of order k, examples of which could well be of independent interest.

Some custom *Mathematica* code was used to determine the two-link failures, which were then visually checked, confirmed, and eliminated. The survivors were checked (at first) completely by hand.

A second method was later used to confirm what was found. The examples for which we had graphical renderings (these are the monotone diagrams on [11] and some of the non-monotones) we converted by hand to Gauss codes. For most of the non-monotones, we had only the *RG encodings* listed on [11] to work with. (Explanations and examples of these will be shown in a later section.) The RG-encodings suffice to inspect the two-component sublinks. Only one of these ("N1") survived this $k = 2$ stage, and fortunately, there was a graphical rendering of that available, so we could continue using Gauss codes.

Having the Gauss codes, we then used some more custom routines, along with the **KnotTheory** package available on the Knot Atlas website [6] to test sublinks for Brunnianism by evaluating their *Jones polynomials.* It suffices to note, for our purposes here, that the Jones polynomial is a *link invariant.* In particular, if the Jones polynomial of a k-component link is distinct from the polynomial of the trivial link of k components (i.e., the link consisting of k unlinked unknots), then the link cannot be trivial. We used this to re-test the 2-component sublinks, then tested the surviving 3-component sublinks, etc., concluding non-Brunnianism whenever a non-trivial Jones polynomial was encountered through $k = 6$.

It must be noted that if the Jones polynomial were the same as that of the trivial link, we would still not be *guaranteed* an unlink (as far we know at present; see Question 3 in section 5), so those cases for which all k-component sublinks yield trivial Jones polynomials for $k = 2$ through 6 would need to be hand-checked. But that did not happen (it would have been cool, though), and as a result, we only had to check just one: the alternating form of M2.

Some minutiae concerning the checking of the monotone 7-Venns is included in what follows in order that our claims are more easily independently verifiable.

4 Alternating simple symmetric 7-Venns

There are 23 monotone simple symmetric Venn diagrams of order 7, six of which have *polar symmetry.* The latter are labeled P1–P6 in [11], the rest are labeled M1–M17. The 33 known non-monotone examples are named N1–N33.

We mark the original seven components with the indices 0 through 6, clockwise. At the stage $k = 2$ (2-component sublinks), it suffices then, by the symmetry of our links, to check the pairs $\{0, 1\}$, $\{0, 2\}$, and $\{0, 3\}$. Let's abbreviate these as 01, 02, 03, and do likewise for other configurations. As an example, the following illustrates that the diagram N5 is not Brunnian, as it fails with the pair 01.

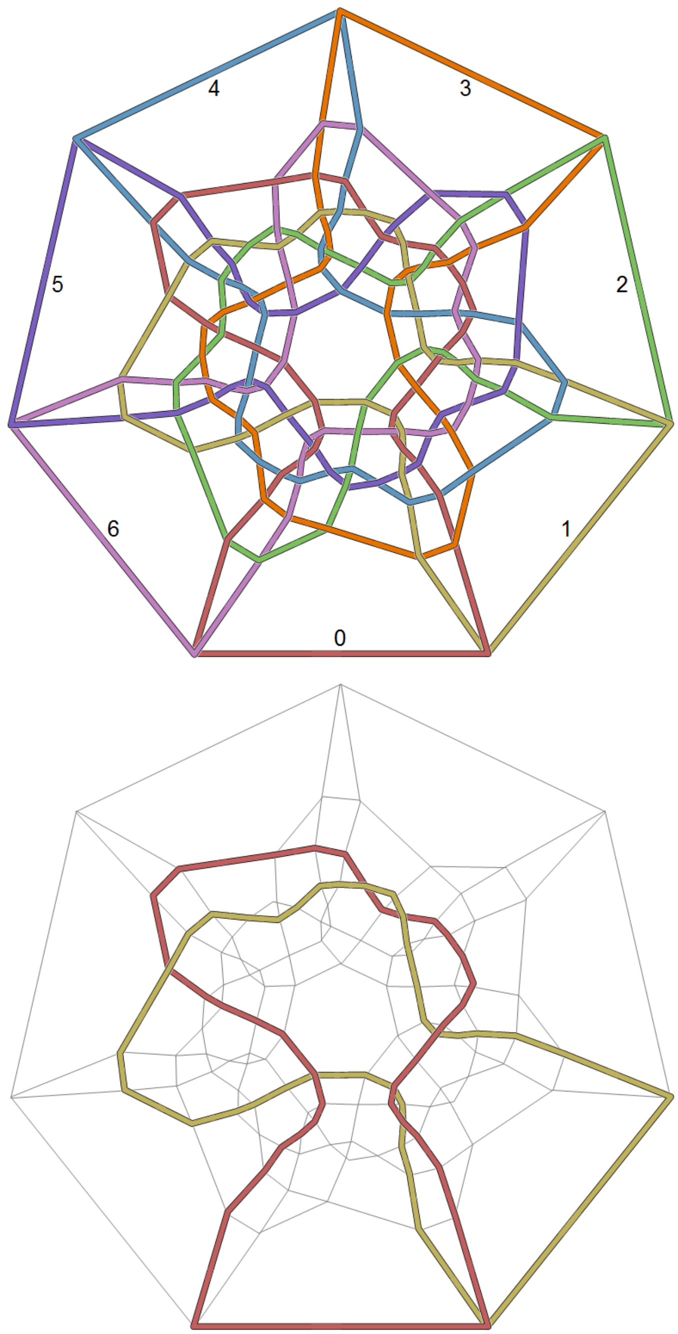

Figure 1: The alternating link on the simple symmetric Venn diagram N5 and its isolated 01 pair, which is linked.

It turns out that 11 of the monotones are thusly ruled out for Brunnianism at the two-component stage, these being M1, M3, M5, M6, M7, M8, M9, M12, M13, M14, M15. Among the non-monotones, N2–N33 are all ruled out! Figure 4 shows the various ways this happens, but first some explanations are in order.

It turns out that M3, M14, M15, N5, N16, N32, and N33 could be called *maximally non-Brunnian*, as they fail for *each* such pair. (The reader can fairly easily check all two-component failures by examining the *RG-encodings*[2] or the "link rendering" images at [11].) Among the monotones, the rest of the two-link failures occur in *exactly two* of the pairs 01, 02, and 03. Though it is possible for three components to be linked in only one pair (simply reverse any one crossing on the Borromean rings for such a thing) it does not happen here, and we offer no explanation for that. Except in the three maximally non-Brunnian cases M3, M14, and M15, the linked pairs above occur in the form of the *Hopf link* (Figure 2). The failures in M3, M14, and M15 each occur as two Hopf links and one "Kramobone" (a link equivalent to the one in Figure 3).

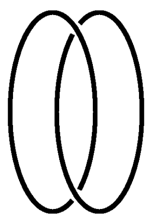

Figure 2: The Hopf link (aka L2a1 and 2^2_1). The simplest nontrivial link and thus also the simplest nontrivial (albeit almost trivial) Brunnian link.

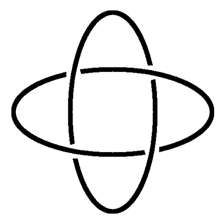

Figure 3: The "Kramobone" (aka L4a1, 4^2_1, among other names [6].)

Among the 33 non-monotones, all fail at the 2-component stage, with the exception of the lone holdout N1. The alternating versions of N5, N16, N32, N33 are maximally non-Brunnian (every pair of components is linked). See [11] for a colored version of this object and a few other of the nonmonotones, including N27—the first discovered example of a non-monotone simple symmetric 7-Venn, which was discovered by none other than Branko Grünbaum [4].

[2]As described in [11], the RG-encoding "is closely related to the Grünbaum encoding: the RG-encoding is obtained by following a curve from the interior/exterior face, numbering and recording the curves in the order that they are encountered." The Grünbaum encoding instead numbers the curves sequentially as one travels around the diagram's outer or inner face. See [1] for a detailed description, in which the diagram P4 is used as the primary example.

	01	02	03
P1	-	-	-
P2	-	-	-
P3	-	-	-
P4	-	-	-
P5	-	-	-
P6	-	-	-
M1	uo	ou	-
M2	-	-	-
M3	uo	uo	ouou
M4	-	-	-
M5	uo	-	ou
M6	-	ou	uo
M7	uo	ou	-
M8	uo	-	ou
M9	uo	ou	-
M10	-	-	-
M11	-	-	-
M12	ou	-	uo
M13	uo	-	ou
M14	uouo	ou	ou
M15	uouo	ou	ou
M16	-	-	-
M17	-	-	-

	01	02	03
N1	-	-	-
N2	ou	uo	-
N3	ouou	-	-
N4	uo	uo	-
N5	ou	ouou	uo
N6	uouo	uouo	-
N7	-	ou	ou
N8	uo	-	uo
N9	uo	-	uo
N10	ou	ou	-
N11	uo	uo	-
N12	ouou	-	-
N13	-	uo	uo
N14	-	ou	ou
N15	uo	uo	-
N16	uo	uouo	ou
N17	uouo	-	-
N18	uo	-	uo
N19	ou	-	ou
N20	-	ou	ou
N21	ou	uo	-
N22	ou	uo	-
N23	uo	uo	-
N24	uo	-	uo
N25	ou	-	ou
N26	ou	ou	-
N27	uo	-	uo
N28	uo	uo	-
N29	ou	uo	-
N30	uo	-	uo
N31	ou	-	ou
N32	uo	uouo	ou
N33	ou	ouou	uo

Figure 4: The over and the under for the 2-component links of the alternating 7-Venns. An “ou” or “uo” designates a Hopf link; “ouou” and “uouo” are Kramobones. The others are unlinked.

Next we’ll meet several varieties of three-component configurations. *So far, all the polar diagrams have passed the 2-component test.*

Question 2 *Is there an obvious reason for that or is it a coincidence?*

At the next level, though, all but one of the polar diagrams are gone: P1 fails for the triple 014; P2 fails for 014; P3 fails for 012 and 014; P5 fails for 012, and P6 fails for 024. Also failing for three components are: M10 for 013; M11 for 024. M16 and M17 each fail for 012. (We note that our symmetry implies that all possible cases of 3-out-of-7 components can be represented by 012, 013, 014, 015, 024.)

	012	013	014	015	024
P1	-	-	B	-	-
P2	-	-	B	-	-
P3	B	-	B	-	-
P4	-	-	-	-	-
P5	B	-	-	-	-
P6	-	-	-	-	B
M2	-	-	-	-	-
M4	-	-	-	-	-
M10	-	B	-	-	-
M11	-	-	-	-	$\star$
M16	B	-	-	-	-
M17	B	-	-	-	-
N1	-	-	-	-	-

Figure 5: The results for the surviving 3-component links of the alternating 7-Venns. The "B" represents the Borromean rings, the $\star$ is something else! The others are unlinked.

It turns out that among these failures for the triples, all but one are the configuration of the Borromean rings. The exception is the failure in M11, which is quite special. We examine it in Section 6.

Meanwhile, P4 survives in the polar region and M2 and M4 are still in the running, as is N1. We are now checking things by hand, so if failures are to occur, we want them to happen at the 4-component stage! Mercifully, the failures of M4 and N1 happen at this stage, each with their 0135 configuration, which is one of the five distinct 4-component sublinks.

This failure—should we really call it that?—is a Brunnian diagram of order four that was discovered by Hermann Brunn himself. A nice rendering of it can be found at [14] and is shown in Figure 6, along with a version obtained from a form of M4 from [11].

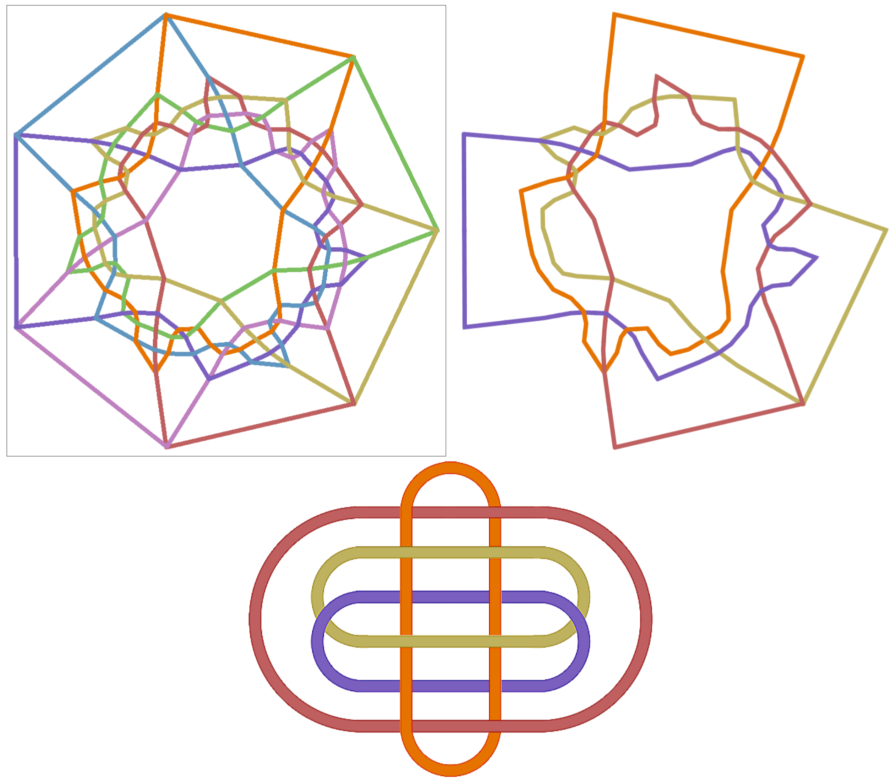

Figure 6: Alternating form of M4; the Brunnian 0135 configuration of same; a simplification based on [14]

.

On to level 5—there are only P4 and M2 remaining; it suffices to check configurations 01234, 01235, 01245. There are no failures, each of these configurations is unlinked! (But it took a while to determine that.)

Level 6—check just 012345. And now here is something: We find that P4 fails with the Brunnian link of order 6 shown in Figure 7.

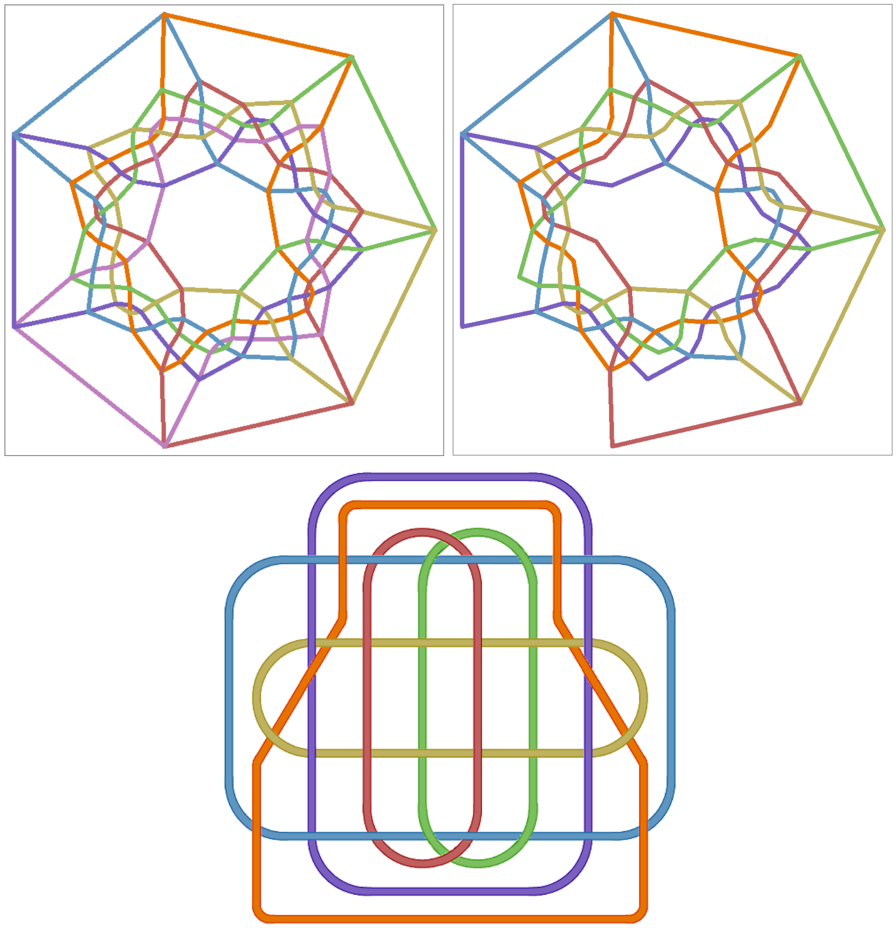

Figure 7: Alternating form of P4; its 012345 Brunnian configuration; our simplification.

And with that, the sole surviving symmetric alternating Venn diagram of order 7 is M2, a symmetric Brunnian link of order 7.

After this strenuous and naive endeavor, we finally took the time to learn a bit more. What would the experts do? While we do not presume to answer that question, we did verify all of the above by computing Jones polynomials using the code available at the KnotAtlas website [6]. (This was not without its own challenges, but that is too long a saga to go into here.)

5 More questions

Question 3 *If a link projects to a union of simple closed curves (topological circles) and has a trivial Jones polynomial, is the link trivial?*

If not, then the link has the same (trivial) Jones polynomial as a trivial link.[3] So far this seems possible, as we alluded to in section 3. There are indeed nontrivial links whose Jones polynomials are trivial, as is dramatically shown in the paper [3]. But in that paper, the examples of such links have some components that are proper knots (i.e., not unknots; not simple closed curves). On the other hand, there exist distinct nontrivial links whose projections are unions of closed curves and whose Jones polynomials are identical, but not trivial. (For instance, different *connected sums* of three Hopf links can be used [10].) So it is natural to ask if there are conditions (such as the one we entertain of unions of topological circles) under which the Jones polynomial can distinguish linked links from unlinks.

Question 4 *Are there more Brunnian links projecting to simple symmetric Venn diagrams?*

By "more" we mean those other than the ones for $n = 2, 3, 5, 7$ already mentioned. And the answer to this question is "Yes."

It turns out that there is only one simple symmetric Venn diagram on 5 sets [11]; its corresponding alternating link turns out to be Brunnian; see Figure 8. If our computations are correct, then the alternating link is the only Brunnian link among the 32 possible (three-dimensionally) symmetric weaves of this Venn diagram.

On the other hand, if the weaves aren't required to be symmetric, then there is exactly one more Brunnian weave on that same Venn diagram, also shown in Figure 8. (The non-symmetry is easy to spot by looking at the components numbered 0 and 4; on one of the figures they do not alternate.) The visually verifiable Brunnianism of the alternating version is discussed in Section 3 of [0]. For the other, it is easy enough to tease apart all the four-component sublinks using a graphics program. (Note that the lack of symmetry means that all five 4-component sublinks need to be checked and found to be unlinked.) To show that the full five-component link is itself linked, try computing any of the known link invariants. The Jones polynomial, for instance, differs from that of the five-component unlink, which settles the matter.

[3] The Jones polynomial of n unlinked components is $(-1-t)^{n-1}$ times a power of t.

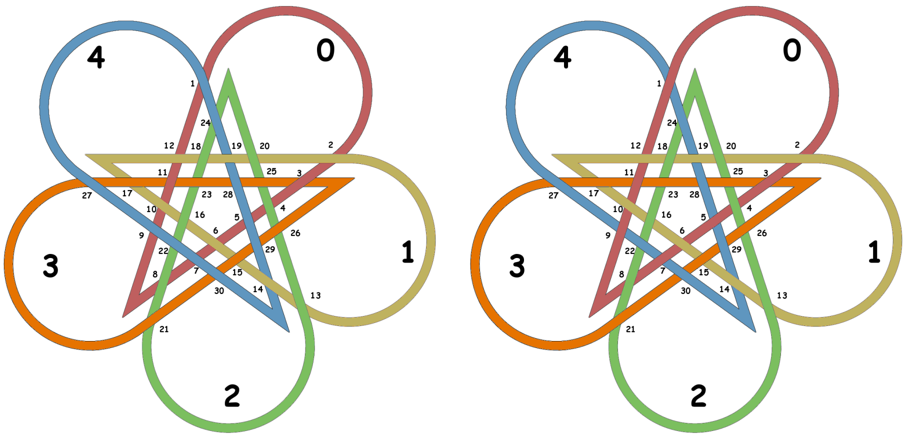

Figure 8: The alternating Brunnian link on the simple symmetric Venn diagram on 5 sets and a non-alternating, non-3D-symmetric version. (Wait, which is which?) They differ at four crossings.

For anyone willing to confirm the claims above, here are glimpses of the "rational" parts of the Jones polynomials we calculated for the two links in Figure 8:

$$-(1+t)^2(1-7t\cdots-+\cdots+146t^6-158t^7+146t^8\cdots-+\cdots-7t^{13}+t^{14})$$
and
$$-1+15t\cdots-+\cdots-932950t^{14}+973374t^{15}-932950t^{16}\cdots-+\cdots+15t^{29}-t^{30}.$$

For the 56 *known* symmetric 7-Venns, there are likewise no more symmetric Brunnian cases, as we mentioned earlier. It is presently unknown if there exist more than the 33 non-monotones listed in [11], though it is conjectured not in [11]: "We believe that these are all of the simple symmetric non-monotone diagrams for $n = 7$." If that turns out to be wrong, then maybe there are also more symmetric alternating Brunnian 7-Venns.

Again, if symmetric weaving is not required, there are more Brunnian examples among the 56 symmetric 7-Venns, probably a great many. By switching small numbers of crossings along one component of the alternating version of M2, for instance, we found this one (the switched crossings are marked):

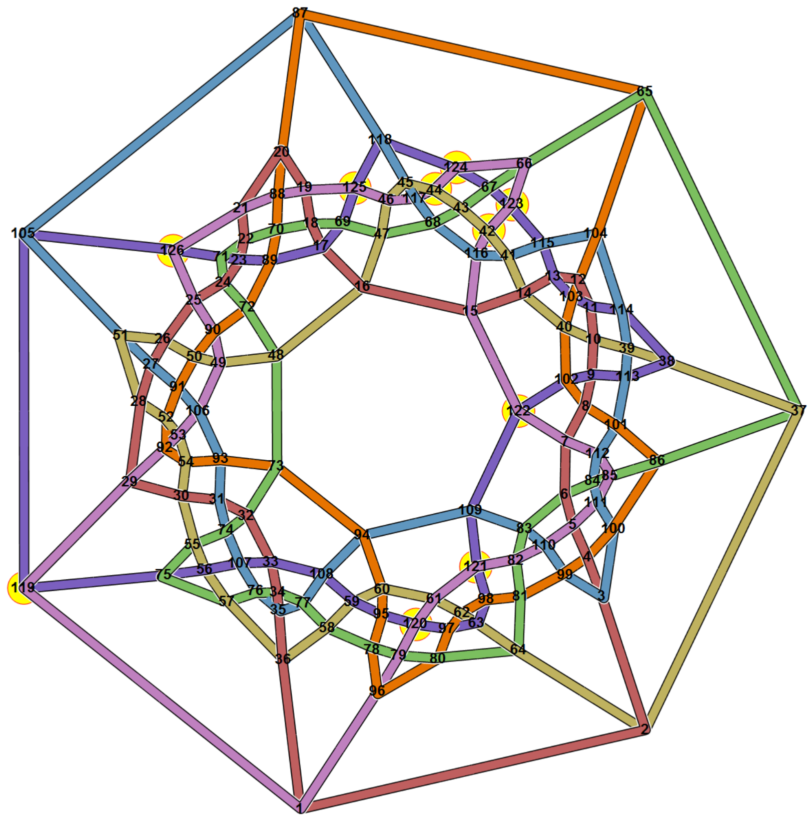

Figure 9: A non-alternating, non-7-periodic, Brunnian weave of M2.

Given the non-symmetric examples above then, we refine the question.

Question 5 *Other than the Hopf link for $n = 2$, the Borromean rings for $n = 3$, the symmetric link above for $n = 5$, and AltM2 for $n = 7$, are there any more three dimensionally symmetric Brunnian links projecting to simple symmetric Venn diagrams?*

The answer might be yes for non-monotone symmetric 7-Venns if more than the presently known 33 cases are ever found to exist. And of course there is a new plethora of simple symmetric 11- and 13-Venns ([8], [9]), though the testing of these could be prohibitively time consuming unless some clever ways to rule out large classes can be found. The tricks of [7] might help.

At the moment, however, our only examples of symmetric Brunnian links on the simple symmetric Venn diagrams are alternating. That prompts the next version of the previous question.

Question 6 *Are there more alternating symmetric Brunnian links on simple symmetric Venn diagrams?*

Again, the answer to the question might be yes for non-monotone symmetric 7-Venns, but only if new ones are discovered. And there might be examples for $n > 7$. Another possibility is that we have made errors in our computing! All of what is presented here needs verification and confirmation. But based on our evidence thus far, the following becomes intriguing. (This really just Question 1 with the Venn part added.)

Question 7 *Must a symmetric Brunnian link on a simple symmetric Venn diagram be alternating?*

And while we have here stuck to symmetric Venn diagrams, there is a bigger universe out there.

Question 8 *What about non-symmetric simple Venn diagrams?*

In [7], a family of Brunnian Venn diagrams on n sets is constructed for all $n \geq 3$. The construction starts with $n = 3$ (the Borromean rings) and recursively constructs Brunnian links that project to a family of simple Venn diagrams constructed by Edwards [2]. These are not alternating links and the diagrams are not symmetric.

Also in [7], all the 5-Venns were considered for possible Brunnianism. Using some braid theory, the authors determined that one of those had no possible Brunnian weaves whatsoever, while others did have Brunnian weaves. Their braid-theoretic notions did not allow them to test the non-monotone cases, but we have done that using brute force. The results of that endeavor, as well as some remarks about Brunnian examples of the 4-Venns will be saved for another time.

6 Marilyn

We said there is something special in M11 — the "failure" at the 3-link stage, in which we get a 3-component linked sublink of the alternating, symmetric M11 Venn diagram. It is Brunnian, but it is not the Borromean rings.

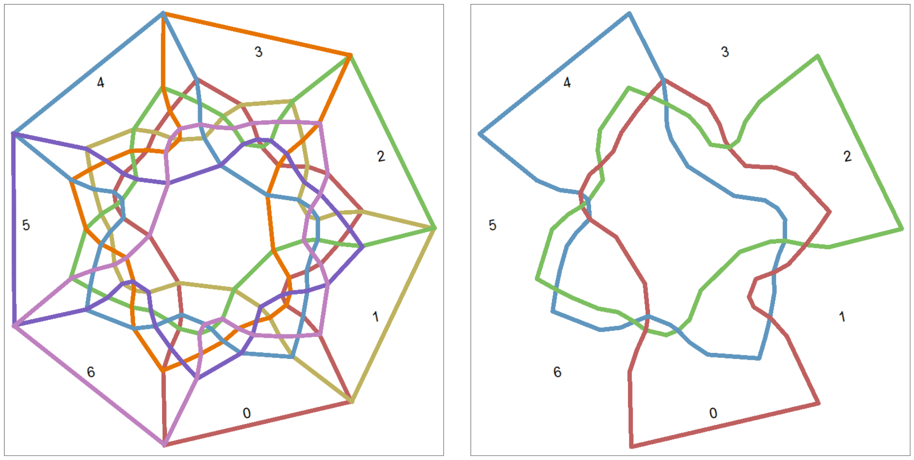

Figure 10: The alternating link on the simple symmetric Venn diagram M11 and its isolated 024 components, which are linked.

When Laura McCormick teased this out of M11, she simplified it in two striking ways:

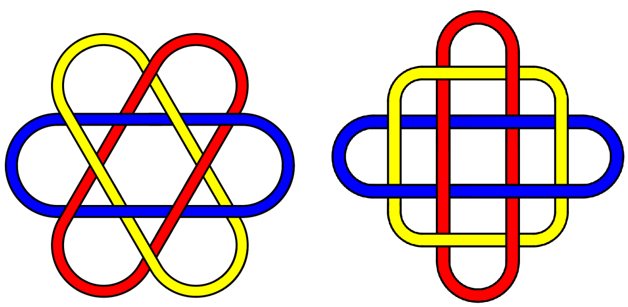

Figure 11: Two simplifications of M11-024 by Laura McCormick.

The right-hand figure reminded me of a cross and the other of the Star of David. This struck me as nicely pseudo-symbolic, a harmonization of two of the world's great religions. At the time, my cousin, Marilyn Henry [16], was fighting a losing battle with cancer. [4] I dubbed the link "Marilyn's Cross" in her honor.

Very shortly thereafter, in a stroke of stunning serendipity, an issue of *Math Horizons* arrived. Marilyn's Cross was on the cover! But it was not Laura's, it was David Swart's [13] sand sculpture!

[4]That is not the correct way to put it! See [5].

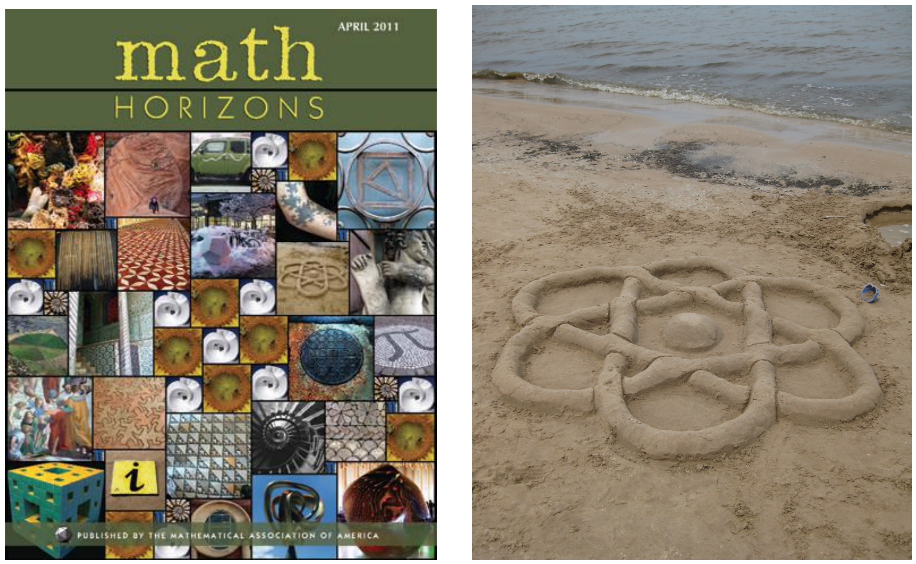

Figure 12: "It is what it is" — photo by David Swart.

After admiring his photo[5], we were in contact with David, who told us of his communication[6] with Dror Bar-Natan about the discovery and the quest to identify it as one of the known Brunnian links. At the time, for some reason, Professor Bar-Natan and company were unable to identify the link, even though they were aware of the link that it indeed turned out to be. Perhaps that was because they were looking only among the known 12-crossing links, which is the form David Swart (and we) had discovered. The link is equivalent to "L10a140" [15],[6] (the 140th alternating link with 10 crossings in the Thistlethwaite Link Table). In any case, the link had a good deal of attention due to all this.

The middle image below is an alternating form (any which must have 10 crossings) of "It Is What It Is" aka "L10a140" aka "M11-024" aka "Marilyn's Cross" etc. This is the same link diagram as the alternating versions at [6][7] I call it the "Hula Hoop" version by Slavik Jablan (1952–2015[8]), who gives it as the second in an infinite sequence of 3-component alternating Brunnian links with $4n + 2$ crossings ($n = 1, 2, 3, \ldots$), the first being the Borromean rings.

[5] See https://www.flickr.com/photos/dmswart/4878510547/

[6] See http://drorbn.net/AcademicPensieve/2010-08/one/A_Link_from_David_Swart.pdf.

[7] See http://katlas.math.toronto.edu/wiki/L10a140.

[8] See http://www.mi.sanu.ac.rs/vismath/jablan.html

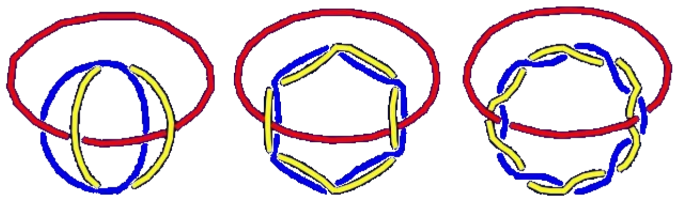

Figure 13: Slavik Jablan's Hula Hoops, $n = 1, 2, 3$.

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